Info

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Contraction map in a metric space

Definition. Contraction map in a metric space

Let $f : X \to X$ be a function on a metric space (X, d). Then f is called a α-**contraction** if there exists $\alpha \in [0, 1)$ such that

$$x, y \in X \implies d(f(x), f(y)) \leq qd(x, y)$$

contractions are uniformly continuous

contractions make Cauchy orbits

Let us have a α-contraction map $f : X \to X$ with

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for $\alpha < 1$.

- Define the orbit sequence of any $x_0 \in X$ inductively such that

$$x_n := f(x_{n-1}), \ \forall n \in \mathbb{N}^+$$

- Now distance between each step keeps getting smaller

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(f(x_n), f(x_{n-1})) \\ &\leq \alpha d(x_n, x_{n-1}) \end{aligned}$$

  by the property of the contraction.
  - Because this is true for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(f(x_n), f(x_{n-1})) \\ &\leq \alpha d(x_n, x_{n-1}) \\ &\leq \alpha(\alpha d(x_{n-1}, x_{n-2})) \\ &\dots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

    by induction.
- Looking at

$$d(x_{n+2}, x_n) \leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x + n)$$

  from the **triangle inequality**, we get the inequality

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \\ &\leq d(x_1, x_0) \sum_{k=n}^{m-1} \alpha^k \\ &= d(x_1, x_0) \left(\frac{\alpha^n}{1-\alpha} - \frac{\alpha^m}{1-\alpha} \right) \\ &< d(x_1, x_0) \frac{\alpha^n}{1-\alpha} \end{aligned}$$

- So for any $\epsilon > 0$, we find a large $N_\epsilon > 0$ such that

$$d(x_1, x_0) \frac{\alpha^{N_\epsilon}}{\alpha - 1} < \epsilon$$

  then

$$d(x_m, x_n) < d(x_1, x_0) \frac{\alpha^{N_\epsilon}}{\alpha - 1} < \epsilon$$

  making $(x_n)_{n\in\mathbb{N}}$ a **Cauchy sequence**.
- If the sequence has a limit $p \in X$ then taking $m \to \infty$ limit in the triangle inequality gives

$$d(p, x_n) \leq d(x_1, x_0) \frac{\alpha^n}{1-\alpha}$$

fixed points

- Given a contraction map $f : X \to X$, if f has two *fixed-points* p, q such that

$$f(q) = q, f(p) = p$$

  then

$$d(p, q) \leq \alpha d(p, q)$$

  which is

$$\iff d(p, q) = 0$$

  which imples $p = q$.
  - So, if the contraction has one fixed point, it has exactly one unique fixed point.

Now because contractions make Cauchy orbits i.e.

$$(f^n(x_0))n_{\in\mathbb{N}}$$

is a Cauchy sequence, we prove the *existence of exactly one fixed point in a complete metric space*:

(Contraction mapping principle) A contraction map

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>
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>
> $$x, y \in X \implies d(f(x), f(y)) \leq qd(x, y)$$

in a *complete* metric space has a **unique fixed point**, that is, $p \in X$ such that

$$f(p) = p$$

and which, for any $x_0 \in X$,

$$\lim_{n\to\infty} f^n(x_0) = p$$

with the estimate

$$d(p, x_n) \leq d(x_1, x_0) \frac{\alpha^n}{1 - \alpha}$$

can contractions exhibit a sensitive dependence on initial conditions?

A contraction map in a metric space **cannot** exhibit a *sensitive dependence on initial conditions*

> **Definition.** Sensitive dependence on initial conditions of dynamics of a map between metric space
>
> Let (X, d) be a metric space and $f : X \to X$ be a map. If there exists a $\epsilon > 0$ such that
>
> $$\forall x \in X, \delta > 0 \left(\exists y \in X, n \in \mathbb{Z}_{>0} \left(\begin{array}{r} d(x, y) < \delta \\ d(f^n(x), f^n(y)) \geq \epsilon \end{array} \right) \right)$$
>
> then dynamics of f is said to **exhibit** ϵ**-sensitive dependence on initial conditions**.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Metric spaces
    - Metric spaces
      - Function between metric spaces
        - Contraction map in a metric space

And it has 6 siblings.

- structures based on sets
  - Metric spaces
    - Metric spaces
      - Function between metric spaces
        - Continuous functions between metric spaces
        - Uniformly continuous functions on a metric space
        - Contraction map in a metric space
        - Topological entropy of maps between metric spaces
        - Limit of a function between metric spaces
        - Sensitive dependence on initial conditions of dynamics of a map between metric space