Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on January 3, 2026 12:15:17 PM,
was last modified on March 26, 2026 5:07:17 PM, and
was exported on September 7, 2026 2:50:38 PM.

# $SO(x^2+y^2-\sqrt{2z^2})(\mathbb{Z}[\sqrt{2}])$

## $SO(x^2+y^2-\sqrt{2z^2})(\mathbb{Z}[\sqrt{2}]) <_{\mathsf{d}} SO(x^2+y^2-\sqrt{2z^2})(\mathbb{R}) \cong SO(1,$

Let

$$q(x,y,z) := x^2+y^2-\sqrt{2}z^2$$

be the quadratic form over $\mathbb{Q}(\sqrt{2})$. Then we have

$$SO(q)(\mathbb{Q}(\sqrt{2})\otimes\mathbb{R}) = SO(q)\times SO(q^\sigma)$$

where

$$(SO(q)\times SO(q^\sigma))(\mathbb{R}) = \underbrace{SO(q)(\mathbb{R})}_{\substack{\cong SO(1,2)(\mathbb{R})\\ \cong PSL(2)(\mathbb{R})}} \times \underbrace{SO(q^\sigma)(\mathbb{R})}_{\cong SO(3)(\mathbb{R})}$$

Then we have a discrete subgroup

$$\begin{aligned}(SO(q)\times SO(q^\sigma))(\mathbb{Z}) &\leq (SO(q)\times SO(q^\sigma))(\mathbb{R})\\ &\cong PSL(2)(\mathbb{R})\times \underbrace{SO(3)(\mathbb{R})}_{\text{compact}}\end{aligned}$$

which is the image of

$$\begin{aligned}SO(q)(\mathbb{Z}[\sqrt{2}]) &\to (SO(q)\times SO(q^\sigma))(\mathbb{R})\\ \gamma &\mapsto (\gamma,\sigma(\gamma))\end{aligned}$$

and we have

$$\Phi : {SL(2)(\mathbb{R})}/{\{I,-I\}} \cong {(SO(q)\times SO(q^\sigma))(\mathbb{R})}/{SO(q^\sigma)(\mathbb{R})}$$

with

$${\Gamma(q)}/{\{I,-I\}} := \Phi^{-1}\left({(SO(q)\times SO(q^\sigma))(\mathbb{Z})}/{SO(q^\sigma)(\mathbb{R})}\right)$$

where $\Gamma(q)$ is the subgroup of $SL(2)(\mathbb{R})$ constructed.

[1]

---

Current note has 0 direct children and 0 total descendants.

- squishy
  - $SO$
    - $SO(x^2+y^2-\sqrt{2z^2})(\mathbb{Z}[\sqrt{2}])$

And it has 5 siblings.

- squishy
  - $SO$
    - $SO(x^2+y^2-\sqrt{2z^2})(\mathbb{Z}[\sqrt{2}])$
    - $SO(2)(\mathbb{R})$
    - $SO^+(2,2)(\mathbb{R})$
    - $SO(3,\mathbb{R})$
    - $SO(4,\mathbb{R})$

---

1. (5.5.4) **Example.** Let $G = \mathrm{SO}(x^2+y^2-\sqrt{2}z^2;\mathbb{R}) \cong \mathrm{SO}(1,2)$. Then $G_{\mathbb{Z}[\sqrt{2}]}$ is a cocompact, arithmetic subgroup of $G$.

   **Proof.** As above, let $\sigma$ be the conjugation on $\mathbb{Q}[\sqrt{2}]$. Let $\Gamma = G_{\mathbb{Z}[\sqrt{2}]}$.
   Let $K' = \mathrm{SO}(x^2+y^2+\sqrt{2}z^2) \cong \mathrm{SO}(3)$, so $\sigma(\Gamma) \subseteq K'$. (However, $\sigma(\Gamma) \not\subseteq G$.) Then, we may

   $$\text{define } \Delta\colon \Gamma \to G\times K' \text{ by } \Delta(\gamma) = (\gamma,\sigma(\gamma)).$$

   Arguing as in the proof of Example 5.5.3 establishes that $\Delta(\Gamma)$ is an arithmetic subgroup of $G\times K'$. (See Exercise 4 for the technical point of verifying that $G\times K'$ is defined over $\mathbb{Q}$.) Since $K'$ is compact, we see, by modding out $K'$, that $\Gamma$ is an arithmetic subgroup of $G$. (This type of example is the reason for including the compact normal subgroup $K'$ in Definition 5.1.19.)
   Let $\gamma$ be any nontrivial element of $\Gamma$. Since $\sigma(\gamma) \in K'$, and compact groups have no nontrivial unipotent elements (see Remark A5.2(2)), we know that $\sigma(\gamma)$ is not unipotent. Therefore, $\sigma(\gamma)$ has some eigenvalue $\lambda \neq 1$. Hence, $\gamma$ has the eigenvalue $\sigma^{-1}(\lambda) \neq 1$, so $\gamma$ is not unipotent. Therefore, Godement's Criterion (5.3.1) implies that $\Gamma$ is cocompact. Alternatively, this conclusion can easily be obtained directly from the Mahler Compactness Criterion (4.4.7) (see Exercise 6). □

   ↩