Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on May 26, 2024 1:23:30 PM,
was last modified on September 7, 2026 2:31:09 PM, and
was exported on September 7, 2026 2:50:39 PM.

$SL(2)(\mathbb{R})$

#wiki/Man/R/3 #Wiki/group/Lie

We understand

$$SL(2)(\mathbb{R}) = Sp(2, \mathbb{R})$$

by definition.

[1]

subgroups and group extensions

$$\begin{tikzcd}
{}^4\mathfrak{gl}(2,\mathbb{R}) & & & {}^4_1GL_+(2,\mathbb{R}) \arrow[r, hook] \arrow[dd, two heads] & {}^4_2GL(2,\mathbb{R}) \arrow[r, "\mathrm{Ad}"] \arrow[dd, two heads] & ? \\
& & \widetilde{SL(2,\mathbb{R})} \arrow[d, two heads, "/\mathbb{Z}"'] & & & \\
{}^3\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{pgl}(2,\mathbb{R}) \arrow[uu, hook] & & {}^3_1SL(2,\mathbb{R}) \arrow[r] \arrow[uur, hook] \arrow[d, two heads, "/\mathbb{Z}_2"'] & {}^3_1?PGL_+(2,\mathbb{R}) \arrow[r, hook, "o?"] & {}^3_2PGL(2,\mathbb{R}) \arrow[r, hook, two heads, "\mathrm{Ad}?", "\cong?"'] & {}^3_2SO(1,2,\mathbb{R}) \\
& & {}^3_1PSL(2,\mathbb{R}) \arrow[rrr, two heads, "\mathrm{Ad}?", "\cong?"'] & & & {}^3_1SO^+(1,2,\mathbb{R}) \arrow[u, hook, "o?"]
\end{tikzcd}$$

subgroups/extensions	quotient
$SL(2)(\mathbb{Z})$	$SL(2)(\mathbb{R})/SL(2)(\mathbb{Z})$
$\{I, -I\}$	$PSL(2)(\mathbb{R})$
$SL(2)(\mathbb{C})$	
discrete subgroups $\iff$ preimage of Fuchsian groups	

types

Given a matrix in $SL(2)(\mathbb{R})$, it has eigenvalues

$$\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm \frac{\sqrt{\mathrm{tr}^2 - 4}}{2}$$

If the discriminant

- If $\mathrm{tr}^2 - 4 < 0$ then

$$\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm i\frac{\sqrt{-\mathrm{tr}^2 + 4}}{2}$$

  which has complex norm 1, thus it is a unit complex number, $\lambda \in U(1)$.
- If $\mathrm{tr}^2 - 4 > 0$, then $\lambda \in \mathbb{R}$.
  - Let $\lambda \in \{-1, 1\}$ then

$$\mathrm{tr} = \lambda + \frac{1}{\lambda} \in \{-2, 2, 0\}$$

    which contradicts $\mathrm{tr}^2 > 4$.
  - Thus $\lambda \in \mathbb{R} \setminus \{-1, 1, 0\}$.

Therefore, we have the following cases:

type	elliptic	parabolic	hyperbolic
$\mathrm{tr}^2 - 4$	< 0	$= 0$	> 0
$\lvert \mathrm{tr} \rvert$	< 2	$= 2$	> 2
eigenvalues λ, λ^{-1}	$\lambda \in U(1) \setminus \{-1, 1\}$ $$\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm i\frac{\sqrt{-\mathrm{tr}}}{2}$$	$$\lambda = \lambda^{-1} = \frac{\mathrm{tr}}{2} \in \{-1,$$	$\lambda \in \mathbb{R} \setminus \{-1, 1, 0\}$ $$\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm \frac{\sqrt{\mathrm{tr}^2 -}}{2}$$
Jordan block in $GL(2)(\mathbb{C})$	$$\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix}$$	$$\pm\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$ for $t \in \mathbb{R}$	$$\pm\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{bmatrix}$$ for $x \in \mathbb{R}, \pm e^x = \lambda$
elements in $SL(2)(\mathbb{R})$ upto conjugacy	conjugate to rotation matrices $R^\theta \in SO(2)(\mathbb{R})$	conjugate to $$\pm\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$ for $t \in \mathbb{R}$	as it has two distinct real eigenvalues so it is conjugate to $$\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$$
$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$	elliptic fixed point at 0	parabolic fixed point at 0	hyperbolic fixed point at 0
$SL(2)(\mathbb{R}) \curvearrowright \boldsymbol{P}(\mathbb{R}^2) =:$	fixes no line in $\mathbb{R}^2$	fixes exactly one line in $\mathbb{R}^2$ $\iff$ fixes exactly $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	fixes exactly two lines in $\mathbb{R}^2$
$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}H^2$	fixes one point $\iff$ conjugate to stabilizer of i which is $SO(2)(\mathbb{R})$	fixes exactly one point in $\partial\mathbb{R}\boldsymbol{H}^n$ $\iff$ conjugate to a transformation only fixing ∞ $\iff$ $$z \mapsto z + t$$	fixes exactly two points in $\partial\mathbb{R}\boldsymbol{H}^n$ $\iff$ conjugate to $$z \mapsto \alpha z$$

exponential map of $SL(2)(\mathbb{R})$

det	eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{-}$	type	Jordan block	exponential	trace of exponential
$\det < 0$	$-x, x \in \mathbb{R} \setminus \{0$	hyperbolic	$\begin{bmatrix} -x & 0 \\ 0 & x \end{bmatrix}$	$\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{bmatrix}$	$e^{-x} + e^{x} > 0$
$\det = 0$	0	parabolic	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$	2
$\det > 0$	eigenvalues $-ix, ix \in i\mathbb{R}$	elliptic	$\begin{bmatrix} -ix & 0 \\ 0 & ix \end{bmatrix}$	$\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix}$	$2\cos(x) \geq -2$

Hence,

$$\operatorname{tr}(\exp(\mathfrak{sl}(2, \mathbb{R}))) \geq -2$$

So for example

Example

The matrix

$$\begin{bmatrix} -2 & \\ & -\frac{1}{2} \end{bmatrix} \in SL(2)(\mathbb{R})$$

is not in the image of the exponential $\mathfrak{sl}(2, \mathbb{R}) \to SL(2)(\mathbb{R})$.

Let there exists $A \in \mathfrak{gl}(2, \mathbb{R})$ with eigenvalues α, β such that

$$\exp(A) = \begin{bmatrix} -2 & \\ & -\frac{1}{2} \end{bmatrix}$$

Then

$$e^{\alpha} = -2, e^{\beta} = -\frac{1}{2}$$

which implies $\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$ but because A is a real 2x2 matrix it's eigenvalues must be roots of a quadratic polynomial so

$$\alpha = \bar{\beta} \implies |e^{\alpha}| = |e^{\beta}|$$

as $|\exp(z)| = e^{\Re(z)}$. But the absolute values do not match $2 \neq \frac{1}{2}$. Thus there is no such $A \in \mathfrak{gl}(2, \mathbb{R})$ either.

generators generating a diffeomorphism

$$\begin{aligned} \mathbb{R} \times \mathbb{R} \times S^1 &\to SL(2)(\mathbb{R}) \\ (x, t, \theta) &\mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} R^{\theta} \end{aligned}$$

is a diffeomorphism.

$$(1, 0, 0) \mapsto I_2$$
$$(1, 0, \pi) \mapsto -I_2$$

If we define

$$X := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$Y := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$R := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

such that

$$\begin{aligned} [X, Y] &= 2Y \\ [X, R] &= -2R - 4Y \\ [Y, R] &= X \end{aligned}$$

so it is not looking that good, but we have

$$\exp(tX) = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}$$
$$\exp(tY) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$
$$\exp(\theta R) = R^{\theta}$$

trace

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} xe^{-t}\sin(\theta) + e^t\cos(\theta) & xe^{-t}\cos(\theta) - e^t\sin(\theta) \\ e^{-t}\sin(\theta) & e^{-t}\cos(\theta) \end{bmatrix}$$

whose trace is

$$xe^{-t}\sin\theta + \cos\theta(e^t + e^{-t})$$

Current note has 0 direct children and 0 total descendants.

- squishy
  - $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$
  - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathbb{U}}$
  - SL
    - $SL(2)(\mathbb{R})$
    - $SL(2)(\mathbb{R}) / SL(2)(\mathbb{Z})$

- $SL(2)(\mathbb{Z})$

And it has 15 siblings.

- squishy
  - SL
    - SL(2)
    - $SL(2)(\mathbb{C})$
    - $PSL(n, \mathbb{C})$
    - $SL(2)(\mathbb{R})$
    - $SL(2)(\mathbb{R})/_{SL(2)(\mathbb{Z})}$
    - H_q
    - $PSL(2)(\mathbb{Z})[2]$
    - $[PSL(2)(\mathbb{Z})[2], PSL(2)(\mathbb{Z})[2$]]
    - $PSL(2)(\mathbb{Z})[N]$
    - $SL(3, \mathbb{R})$
    - $SL(2)(\mathbb{Z})$
    - $PSL(2)(\mathbb{Z})$
    - Congruent subgroups of $PSL(2)(\mathbb{Z})$
    - $SL(n, \mathbb{Z})$
    - $SL(2)(\mathbb{Z}_2)$

---

1. SL(2,R).pdf ↩
2. webspace.science.uu.nl/~ban00101/lecnotes/sltwoR.pdf ↩