ⓘ Info

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$\widehat{\ }: L^2(0,\infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathrm{U}})$

Definition.

$$\mathscr{O}^p(H^2_{\mathrm{U}}) := \left\{ f \in \mathscr{O} \,\middle|\, \sup_{y\in(0,\infty)} \int_{\mathbb{R}+iy} |f|^p < \infty \right\}$$

[1]

Let $f \in L^2(0,\infty) \leq L^2(\mathbb{R})$ and $z \in H^2_{\mathrm{U}}$ consider

$$\begin{aligned} |f(t)\exp(itz)| &= \left|f(t)e^{-t\Im(z)}\right| \\ \implies t \mapsto f(t)\exp(itz) &\in L^2(0,\infty) \end{aligned}$$

so we have the Fourier transform

$$\hat{f}(z) := \int_{(0,\infty)} f(t)\exp(itz)$$

for all $z \in H^2_{\mathrm{U}}$.

(Paley-Wiener)

$$\hat{\ }: L^2(0,\infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathrm{U}})$$

[2]

For any $\delta > 0$ and $y \geq \delta$ we have

$$\begin{aligned} \left| \int_{x\in(0,\infty)} f(x)e^{2\pi i x z} \right| &\leq \int_{x\in(0,\infty)} |f(x)|e^{2\pi x|z|} \\ &\leq \sqrt{\int_{(0,\infty)} |f|^2} \sqrt{\int_{x\in(0,\infty)} e^{-4\pi x\delta}} \end{aligned}$$

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \mathscr{O}^2(H^2_{\mathrm{U}}) \arrow[d, "\Im(z)\to 0"] \\ L^2(0,\infty) \arrow[ur, "\widehat{\ }"] \arrow[r, hook, "\widehat{\ }"] & L^2(\mathbb{R}) \end{tikzcd}$$

$$\begin{tikzcd} & \mathscr{O}^2(H^2_{\mathrm{U}}) \arrow[d, "\Im(z)\to 0"] \\ L^2(0,\infty) \arrow[r, "\hat{\ }"] & L^2(\mathbb{R}) \end{tikzcd}$$

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1. book.pdf ↩
2. **Definition.**

$$\mathscr{O}^p(H^2_{\mathrm{U}}) := \left\{ f \in \mathscr{O} \,\middle|\, \sup_{y\in(0,\infty)} \int_{\mathbb{R}+iy} |f|^p < \infty \right\}$$

↩