Info

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local basis of a tangent space of a smooth manifold

can we have tangents for arbitary velocities in a chart?

Given a local chart (U, φ) on a smooth manifold M,

- $$t \mapsto \varphi^{-1}(\varphi(p) + t\mathbf{v})$$
  is a smooth curve on M which is at p at $t = 0$.
- $$(- \circ \varphi^{-1}(\varphi(p) + t\mathbf{v})'(0)$$
  is an element of T_pM which is the velocity of the previous curve.
  - Inserting $f \in \mathcal{C}^\infty(M)$ inside it gives
    $$\underbrace{(f \circ \varphi^{-1}(\varphi(p) + t\mathbf{v})'}_{\mathbb{R}\to\mathbb{R}}(0) = \partial_\mathbf{v} \underbrace{(f \circ \varphi^{-1})}_{\mathbb{R}^n\to\mathbb{R}}(p)$$
    by the usual chain rule in $\mathbb{R}^n$.
- Here, we see we have an linear bijection
  $$\begin{aligned} \mathbb{R}^n &\to T_pM \\ \mathbf{v} &\mapsto (- \circ \varphi^{-1}(\varphi(p) + t\mathbf{v})'(0) \end{aligned}$$
  which seems to be the inverse of

  > For a n-manifold M, map
  > $$\begin{aligned} T_pM &\to \mathbb{R}^n \\ (- \circ \gamma)'(0) &\mapsto (\varphi \circ \gamma)'(0) \end{aligned}$$
  > is an $\mathbb{R}$-linear isomorphism for any local chart φ at p. Hence,
  > $$\dim(T_pM) = \dim(M).$$

basis of tangent space for a given chart

Definition. Local basis of T_pM for a given chart near $p \in M$

Let a local chart (U, x) near $p \in M$ is made up of

$$x = (x_1, \dots, x_n),\ x_i \in \mathcal{C}^\infty(U)$$

- Putting standard basis $\hat{e}_i$ of $\mathbb{R}^n$
  $$t \mapsto \varphi^{-1}(x(p) + t\hat{e}_i)$$
  is a smooth curve for all $1 \le i \le n$.
- We define an element of T_pM by
  $$\frac{\partial}{\partial x_i}(-)_p := (- \circ x^{-1}(x(p) + t\hat{e}_i))'(0)$$
  because $\frac{\partial}{\partial x_i}(f)_p = (f \circ x^{-1}(\varphi(p) + t\hat{e}_i))'(0)$ which equals $= \frac{\partial}{\partial x_i}(f \circ x^{-1})(p)$ by chain rule in $\mathbb{R}^n$.

> $$\frac{\partial}{\partial x_i} x_j = \delta_{ij}$$
>
> $$\begin{aligned} &\frac{\partial}{\partial x_i} x_j \\ &= ((x_j \circ x^{-1}(\varphi(p) + t\hat{e}_i))'(0)) \\ &= \partial_{e_i}(x_j \circ x^{-1}) \\ &= \partial_{e_i} \underbrace{(x \circ x^{-1})_j}_{\mathsf{Id}_j} \\ &= \delta_{ij} \end{aligned}$$

- For any smooth γ such that $\gamma(0) = p$ we ask the question
  $$(- \circ \gamma)'(0) = \sum_{i=1}^{\dim M} (?)_i \frac{\partial}{\partial x_i}(-)_p$$
  - Putting
    > $$\frac{\partial}{\partial x_i} x_j = \delta_{ij}$$

$$\begin{aligned}\frac{\partial}{\partial x_i}x_j \\ &= ((x_j \circ x^{-1}(\varphi(p) + t\hat{e}_i))'(0)) \\ &= \partial_{e_i}(x_j \circ x^{-1}) \\ &= \partial_{e_i}\underbrace{(x \circ x^{-1})_j}_{\mathrm{Id}_j} \\ &= \delta_{ij}\end{aligned}$$

we get

$$(x_i \circ \gamma)'(0) = \sum_i^{\dim M} (?)_i \frac{\partial}{\partial x_i}(x_i)_p = (?)_i$$

- Hence,

$$(- \circ \gamma)'(0) = \sum_{i=1}^{\dim M} (x_i \circ \gamma)'(0) \frac{\partial}{\partial x_i}(-)_p$$

Thus, we obtain a local basis of T_pM

$$\left(\frac{\partial}{\partial x_i}(-)\Big|_p\right)_{i=1}^{\dim M}$$

change of basis

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^{\dim M} \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - local basis of a tangent space of a smooth manifold

And it has 8 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - Canonical bijection of T_pM with equivalence classes of velocities
          - Tangents on smooth manifolds as derivations $\mathcal{C}^\infty(M) \to \mathbb{R}$
          - local basis of a tangent space of a smooth manifold
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