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Embedding smooth manifolds in $\mathbb{R}^n$

(Whitney embedding theorem) Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

- Let M be a compact smooth manifold. In this case M admits a finite cover

  $$\{B_1, \ldots, B_m\}$$

  in which each B_i is a *regular* coordinate ball that is there is a smooth coordinate map

  $$\varphi_i : B_i' \supseteq \bar{B}_i \to \mathbb{R}^n$$

  which restricts to a diffeomorphism from $\bar{B}_i$ to a compact subset of $\mathbb{R}^n$.

Theorem 6.15 (Whitney Embedding Theorem). *Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Proof. Let M be a smooth n-manifold with or without boundary. By Lemma 6.14, it suffices to show that M admits a smooth embedding into some Euclidean space.

First suppose M is compact. In this case M admits a finite cover $\{B_1, \ldots, B_m\}$ in which each B_i is a regular coordinate ball or half-ball. This means that for each i there exist a coordinate ball $B_i' \supseteq \bar{B}_i$ and a smooth coordinate map $\varphi_i : B_i' \to \mathbb{R}^n$ that restricts to a diffeomorphism from $\bar{B}_i$ to a compact subset of $\mathbb{R}^n$. For each i, let $\rho_i : M \to \mathbb{R}$ be a smooth bump function that is equal to 1 on $\bar{B}_i$ and supported in B_i'. Define a smooth map $F : M \to \mathbb{R}^{nm+m}$ by

$$F(p) = \big(\rho_1(p)\varphi_1(p), \ldots, \rho_m(p)\varphi_m(p), \rho_1(p), \ldots, \rho_m(p)\big),$$

where, as usual, $\rho_i\varphi_i$ is extended to be zero away from the support of ρ_i. We will show that F is an injective smooth immersion; because M is compact, this implies that F is a smooth embedding.

To see that F is injective, suppose $F(p) = F(q)$. Because the sets B_i cover M, there is some i such that $p \in B_i$. Then $\rho_i(p) = 1$, and the fact that $\rho_i(q) = \rho_i(p) = 1$ implies that $q \in \operatorname{supp} \rho_i \subseteq B_i'$, and

$$\varphi_i(q) = \rho_i(q)\varphi_i(q) = \rho_i(p)\varphi_i(p) = \varphi_i(p).$$

Since φ_i is injective on B_i', it follows that $p = q$.

Next, to see that F is a smooth immersion, let $p \in M$ be arbitrary and choose i such that $p \in B_i$. Because $\rho_i \equiv 1$ on a neighborhood of p, we have $d(\rho_i\varphi_i)_p = d(\varphi_i)_p$, which is injective. It follows easily that dF_p is injective. Thus, F is an injective smooth immersion and hence an embedding.

Now suppose M is noncompact. Let $f : M \to \mathbb{R}$ be a smooth exhaustion function. Sard's theorem shows that for each nonnegative integer i, there are regular values a_i, b_i of f such that $i < a_i < b_i < i + 1$. Define subsets $D_i, E_i \subseteq M$ by

$$D_0 = f^{-1}\big((-\infty, 1]\big), \quad E_0 = f^{-1}\big((-\infty, a_1]\big);$$
$$D_i = f^{-1}\big([i, i+1]\big), \quad E_i = f^{-1}\big([b_{i-i}, a_{i+1}]\big), \quad i \geq 1.$$

By Proposition 5.47, each E_i is a compact regular domain. We have $D_i \subseteq \operatorname{Int} E_i$, $M = \bigcup_i D_i$, and $E_i \cap E_j = \varnothing$ unless $j = i-1$, i, or $i+1$. The first part of the proof shows that for each i there is a smooth embedding of E_i into some Euclidean space, and therefore by Lemma 6.14 there is an embedding $\varphi_i : E_i \to \mathbb{R}^{2n+1}$. For each i, let $\rho_i : M \to \mathbb{R}$ be a smooth bump function that is equal to 1 on a neighborhood of D_i and supported in $\operatorname{Int} E_i$, and define $F : M \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$ by

$$F(p) = \left(\sum_{i \text{ even}} \rho_i(p)\varphi_i(p), \sum_{i \text{ odd}} \rho_i(p)\varphi_i(p), f(p)\right).$$

Then F is smooth because only one term in each sum is nonzero in a neighborhood of each point, and F is proper because f is. We will show that F is also an injective smooth immersion, hence a smooth embedding.
