Info

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Hamiltonian vector fields on a symplectic manifold

Definition. Hamiltonian vector fields on a symplectic manifold

Let (M, ω) be a symplectic manifold and let $H \in \mathcal{C}^\infty(M)$. Then $\mathrm{d}H$ is a 1-form. Define the *unique* vector field $X_H \in \Gamma(TM)$ such that

$$-\omega(X_H, -) = \mathrm{d}H$$

as the **Hamiltonian vector field** with H, also denoted by

$$\omega^{-1}(-\mathrm{d}H)$$

$$\begin{aligned}\omega^{-1}(-\mathrm{d}(-)) : \mathcal{C}^\infty(M) &\to \mathrm{SympVec}(M,\omega) \leq_{\mathsf{LieAlg}} \mathrm{Vec}(M)\\ f &\mapsto \omega^{-1}(-\mathrm{d}f)\end{aligned}$$

is a Lie algebra homomorphism.

Assuming $\omega^{-1}(-\mathrm{d}H)$ is complete, let

$$\exp(t\omega^{-1}(-\mathrm{d}H))$$

be the $\mathbb{R}^1$-parameter family of diffeomorphisms generated (flow) by $\omega^{-1}(-\mathrm{d}H)$

Each diffeomorphism $\exp(t\omega^{-1}(-\mathrm{d}H)), t \in \mathbb{R}$ preserves ω, that is

$$t \in \mathbb{R} \implies \exp(t\omega^{-1}(-\mathrm{d}H))^*\omega = \omega$$

Let, $\rho^t := \exp(t\omega^{-1}(-\mathrm{d}H))$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^H)^*\omega &= (\rho_t^H)^*\mathcal{L}_{X_H}\omega\\ = (\rho_t^H)^*(\underbrace{\omega X_H}_{\mathrm{d}H} + \underbrace{\mathrm{d}\omega}_{0}) &= 0\end{aligned}$$

Therefore, every function on (M, ω) gives a family of *symplectomorphisms*.

Notice how the proof involved both the non-degeneracy and the closedness of ω.

flows

Flow

$$\exp(tX_H)$$

of a

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is a symplectic map for each t.

- Preserves ω^n volume form
  - No "dissipation"
  - No attractors!

not all symplectic vector fields are Hamiltonian

Example

the symplectic vector field on $\mathbb{R}^2 \setminus \{0\}$ which is *not* Hamiltonian

We want the vector field $X = X_1\partial_1 + X_2\partial_2$ such that

$$\begin{aligned}\mathrm{d}x \wedge \mathrm{d}y(X_1\partial_1 + X_2\partial_2, -) &= \tfrac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}(-)\\ X_1\mathrm{d}x \wedge \mathrm{d}y(\partial_1, -) + X_2\mathrm{d}x \wedge \mathrm{d}y(\partial_2, -) &= \tfrac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}(-)\\ X_1 &= \tfrac{x}{x^2+y^2}\\ -X_2 &= -\tfrac{y}{x^2+y^2}\end{aligned}$$

Thus we have

$$\frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{x^2+y^2}(x\partial_1 + y\partial_2)$$

$$\begin{array}{ccccccc} \mathcal{C}^\infty(M) & \longrightarrow & \text{exact } \Omega^1(M) & \leq & \text{closed } \Omega^1(M) & \leq & \Omega^1(M) \\ H & & \mathrm{d}H & & \alpha & & \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ & & \mathrm{HamVec}(M,\omega) & \leq & \mathrm{SymVec}(M,\omega) & \leq & \mathrm{Vec}(M) \\ & & -\omega^{-1}\mathrm{d}H & & -\omega^{-1}\alpha & & \end{array}$$

> **Proposition:** For a symplectic manifold (M,ω),
>
> $$\frac{\mathrm{SymVec}(M,\omega)}{\mathrm{HamVec}(M,\omega)} \cong H^1(M,\mathbb{R})$$

commutator of symplectic vector fields

> **Proposition:** Let (M,ω) be a symplectic manifold and $X, Y \in \mathrm{Vec}(M)$. Then the Hamiltonian vector field of $\omega(X,Y)$ is $[X,Y]$.

$$\omega([X,Y],-) =? -\mathrm{d}(\omega(X,Y))(-)$$

category

Definition. Category of Hamiltonian flows on symplectic manifolds

HamilSymp

- **Objects:** (M,ω,H) where (M,ω) is a symplectic manifold and $H \in \mathcal{C}^\infty(M)$
- **Morphisms:** $\phi : M_1 \to M_2$ smooth map such that

$$\phi^*\omega_2 = \omega_1, \phi^* H_2 = H_1$$

infinitesimal HamilSymp morphisms

Definition. infinitesimal **HamilSymp** morphisms

A **infinitesimal Hamiltonian morphism** on (M,ω,H) is a vector field $V \in \mathrm{Vec}(M)$ whose flow preserves ω and H, that is

$$\mathcal{L}_V\omega = 0, \mathcal{L}_V H = 0$$

so they are precisely those **symplectic vector fields** V which satisfy

$$VH = 0 \iff H \in \ker V \iff V \in \ker \mathrm{d}H$$

The vector space of all such vector fields is

$$\mathrm{SymVec}(M,\omega,H) := \ker \mathrm{d}H \cap \mathrm{SymVec}(M,\omega)$$

> **Proposition:** Let (M,ω,H) is a Hamiltonian flow.
>
> - **(constants produces infinitesimal symmetries)** Let $f \in \mathcal{C}^\infty(M)$ is *constant* along the Hamiltonian flow of H. Then the Hamiltonian vector field of f is an *infinitesimal Hamiltonian morphism* for H
>
> $$X_H f = 0 \implies X_f H = 0$$
>
>   - Thus
>
> $$\begin{aligned} \ker X_H &\to \mathrm{SymVec}(M,\omega,H) \\ f &\mapsto -\omega^{-1}\mathrm{d}f \end{aligned}$$
>
> - **(infinitesimal symmetries only produce local constants)** Let V is an *infinitesimal Hamiltonian morphism* for H then $\omega(V,-)$ is closed as V is symplectic. But $\omega(V,-)$ is exact $\iff$ there is a $f \in \ker X_H$ such that V is the Hamiltonian vector field of f.
>   - Thus, every symmetry produces a constant of the Hamiltonian flow given $H^1(M) = 0$.
>   - Thus, locally, every symmetry produces a constant of the Hamiltonian flow.

Current note has 5 direct children and 5 total descendants.

And it has 8 siblings.