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Holomorphic functions on punched disk

$$z_0 + RD^* := \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$$

- Because [modulus of non-constant holomorphic functions cannot attain its maximum in the interior of its domain](#), holomorphic function needs to have its maximum at the boundary or at z_0 .
- Can it have maximum at z_0 ? No!

expressing as a Laurent series

Proposition: Let $z_0 + RD^* := \{z : 0 < |z - z_0| < r\}$ and

$$f : z_0 + RD^* \rightarrow \mathbb{C}$$

be holomorphic, then it has a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

which converges in $a + RD^*$ (that means each of the series $\sum_{n \geq 0} a_n z^n, \sum_{n \geq 1} a_{-n} z^{-n}$ converges near $z = 0$) given by

$$a_n = \frac{1}{2\pi i} \int_{S^1_s} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

singularities

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Then there are two cases:

- $n < -k \implies c_n$. Let k be the largest such k , then
 - if $k \geq 0$, the function is analytic, *removable singularity*
 - if $k < 0$, f has a *pole*
 - $\{n : c_n \neq 0\}$ is not bounded below, *essential singularity*
- Let

$$\{z : |f(z) - A| < \epsilon\} = \emptyset$$

for some $A \in \mathbb{C}, \epsilon > 0$

- Then

$$|f(z) - A| \geq \epsilon \text{ on } U^*$$

which means

$$h(z) := \frac{1}{f(z) - A} \text{ on } U^*$$

is holomorphic and

$$|h(z)| = \left| \frac{1}{f(z) - A} \right| \leq \frac{1}{\epsilon} \text{ on } U^*$$

which means a is a removable singularity of $h(z)$

- Then

$$f(z) = A + \frac{1}{h(z)}$$

is holomorphic which means f cannot have an essential singularity on U^*

- This means if f has an essential singularity, it's image is dense in $\mathbb{C}$

Types

f is holomorphic on $z_0 + RD^*$	type of (isolated) singularities	the Laurent series $f(z) = \sum_{n \in \mathbb{Z}} c_n (z - a)^n$	image of f on $z_0 + \mathbb{D}_R^*$
	removable singularity, holomorphic at z_0	$c_n = 0$ for all $n < 0$	bounded
$(z - z_0)^k f(z)$ has a <i>removable singularity</i> for some $k > 0$ but f does not	pole, meromorphic at z_0	eventually $c_n = 0$	$\ f(z)\ \rightarrow \infty$ as $\ z\ \rightarrow 0$
$(z - z_0)^k f(z)$ does not have a removable singularity for any $k \in \mathbb{N}$	essential singularity	c_n does not become 0 eventually	image is dense on $\mathbb{C}$ (actually reaches almost all values on $\mathbb{C}$)

The function

$$e^{1/z}$$

does not reach the point 0

 Warning

Just because some function has Laurent series, does not mean *residue theorem* works. It works only for isolated singularities, that is, locally must be a function on punched disk.

Current note has 0 direct children and 0 total descendants.

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 - [Holomorphic functions on spaces over \$\mathbb{C}\$ of dimension 1](#)
 - [mapping](#)
 - [Holomorphic functions on punched disk](#)

And it has 13 siblings.

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 - [Branch points and ramification index of holomorphic functions](#)
 - [Disk in holomorphic image of disk](#)
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 - [Maximum and minimum of holomorphic functions](#)
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