ⓘ Info

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$\mathcal{C}_0[0,1]$

It does not make sense to talk about boundary values for L^p spaces because they are defined upto equality almost everywhere.

So we consider the subspace of continuous functions on $[0,1]$

$$\mathcal{C}_0[0,1] := \{f \in \mathcal{C}[0,1] \mid f(0) = 0 = f(1)\}$$

- Its $\ker \mathrm{eval}_0 \cap \mathcal{C}[0,1] \cap \ker \mathrm{eval}_1$.
- $$\mathcal{C}[0,1]_{0,0} \hookrightarrow \mathcal{C}(S^1) \hookrightarrow \mathcal{C}[0,1]$$
- 0 is the only constant function in the domain.

integration

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$$\int_{[0,1]} : \mathcal{C}[0,1]_{0,0} \to \mathbb{C}$$

integral

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$$\int_{[0,\bullet]} : \mathcal{C}[0,1]_{0,0} \to \mathcal{C}[0,1]_{0,-}$$

- The operator has no eigenvalues as

$$\begin{aligned}\int_{[0,x]} f &= \lambda f \\ \implies f &= \lambda f', f(0) = 0 \\ \implies f &= 0\end{aligned}$$

- The "inverse" operator is derivative

$$\mathfrak{D} : \mathcal{C}^1[0,1]_{0,0} \to \mathcal{C}[0,1]$$

- $$\begin{aligned}\left|\int_{[0,x]} f\right| &\leq \int_{[0,x]} |f| \\ &\leq \|f\|_\infty \int_{[0,x]} 1 \\ &\leq \|f\|_\infty \\ \left\|\int_{[0,\bullet]} f\right\|_\infty &\leq \|f\|_\infty\end{aligned}$$

derivative

$$\mathfrak{D} : \mathcal{C}^1[0,1]_{0,0} \to \mathcal{C}[0,1]$$

double derivative and its domain

Definition. Space of continuously double differentiable functions on $[0,1]$ with boundary value 0

We consider our domain to be

$$\mathcal{D}(\mathfrak{D}^2) := \{f \in \mathcal{C}^2([0,1],\mathbb{C}) \mid f(0) = 0 = f(1)\}$$

which is a subspace of $L^2[0,1]$. On this domain we have the operator

$$\mathfrak{D}^2 : \mathcal{D}(\mathfrak{D}^2) < L^2[0,1] \to \mathcal{C}[0,1] < L^2[0,1]$$

- The domain is $\ker \mathrm{eval}_0 \cap \mathcal{C}^2[0,1] \cap \ker \mathrm{eval}_1$.
- $0 \in \mathcal{D}^2(\mathfrak{D})$ is the only constant function in this domain.

> **Proposition:** For
>
> **Definition.** Space of continuously double differentiable functions on $[0,1]$ with boundary value 0
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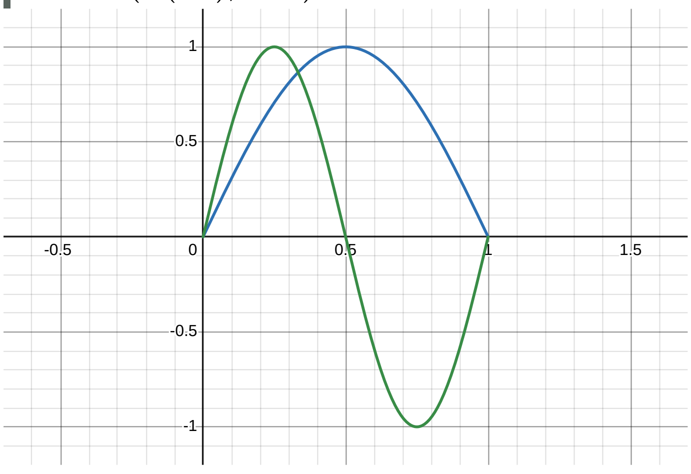
> $$\mathfrak{D}^2 : \mathcal{D}(\mathfrak{D}^2) < L^2[0,1] \to \mathcal{C}[0,1] < L^2[0,1]$$
>
> $\mathcal{D}(\mathfrak{D}^2)$ is dense in $L^2[0,1]$. $\mathfrak{D}^2$ is a linear but not a bounded operator on $\mathcal{D}(\mathfrak{D}^2)$.

> **Proposition:** The set of functions
>
> $$\{\sin(n\pi x) \mid n \in \mathbb{Z}_{>0}\}$$

> a set of orthogonal eigenbasis of $L^2[0,1]$. Moreover, they are an orthogonal eigenbasis for $\mathfrak{D}^2$ on $(\mathcal{D}(\mathfrak{D}^2), \|\ \|_2)$.



- For $n \in \mathbb{Z}_{>0}$

$$\|\sin n\pi x\|_2^2 = \int_{[0,1]} (\sin(n\pi x))^2 = \frac{1}{2}$$

- However, this basis is not an eigenbasis of $\mathfrak{D}$ as

$$\mathfrak{D}(\sin n\pi x) = (n\pi)\cos(n\pi x)$$

- In this basis,

$$\mathfrak{D}^2 = \begin{bmatrix} -\pi^2 & & & & \\ & -4\pi^2 & & & \\ & & \ddots & & \\ & & & -(k\pi)^2 & \\ & & & & \ddots \end{bmatrix}$$

  which easily seen to be not a bounded operator.

exponentiating the double derivative

> Question
>
> We want to solve the initial and boundary value problem on $[0,1]$ of the **Heat equation**
>
> - Solve for a function $u(x,t)$ (regular enough)
>
> $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ on } (x,t) \in [0,1]\times(0,\infty)$$
>
> - for given *initial value*
>
> $$x \in \mathbb{R} \implies u(x,0) = f(x)$$
>
>   for some $f \in L^2[0,1]$
> - and *boundary conditions*
>
> $$t \in \mathbb{R} \implies u(0,t) = 0 = u(1,t)$$

From

> - In this basis,
>
> $$\mathfrak{D}^2 = \begin{bmatrix} -\pi^2 & & & & \\ & -4\pi^2 & & & \\ & & \ddots & & \\ & & & -(k\pi)^2 & \\ & & & & \ddots \end{bmatrix}$$
>
>   which easily seen to be not a bounded operator.

we can define the exponential

$$\exp(t\mathfrak{D}^2) := \begin{bmatrix} e^{-t(\pi^2)} & & & & \\ & e^{-t(4\pi^2)} & & & \\ & & \ddots & & \\ & & & e^{-t(k\pi)^2} & \\ & & & & \ddots \end{bmatrix}$$

that is

> Definition. Exponential of $\mathfrak{D}^2$
>
> For any $t \in (0,\infty)$ the exponential of $\mathfrak{D}^2$ on its domain with boundary values 0
>
> $$\exp(t\mathfrak{D}^2) : \mathcal{D}(\mathfrak{D}^2) \to \mathcal{D}(\mathfrak{D}^2)$$
> $$\exp(t\mathfrak{D}^2)\left(\sum_{n\in\mathbb{Z}_{>0}} a_n \sin n\pi x\right) := \sum_{n\in\mathbb{Z}_{>0}} a_n e^{-t(n\pi)^2}\sin(n\pi x)$$

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>
> is a bounded operator, thus has an **unique bounded extension**
>
> $$\exp(t\mathfrak{D}^2) : L^2[0,1] \to L^2[0,1]$$
>
> as $\mathcal{D}(\mathfrak{D}^2) < L^2[0,1]$ is dense and $L^2[0,1]$ is complete.

Because for each $t > 0$ there is a exponential factor $a_n e^{-(n\pi)^2 t}$ the sine-Fourier series actually uniformly converges to a very nice function!

As $(\sin(n\pi x))/\sqrt{2}$ is a orthonormal basis and let $f \in L^2[0,1]$

$$f = \sum_{n\geq 0} \sqrt{2} a_n \frac{\sin(n\pi x)}{\sqrt{2}}$$

So (a_n) is a square integrable sequence, so it must be bounded.

For $f \in L^2[0,1]$, for every $t > 0$

Definition. Exponential of $\mathfrak{D}^2$

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$$\exp(t\mathfrak{D}^2) : \mathcal{D}(\mathfrak{D}^2) \to \mathcal{D}(\mathfrak{D}^2)$$
$$\exp(t\mathfrak{D}^2)\left(\sum_{n\in\mathbb{Z}_{>0}} a_n \sin n\pi x\right) := \sum_{n\in\mathbb{Z}_{>0}} a_n e^{-t(n\pi)^2} \sin(n\pi x)$$

converges uniformly to a **smooth** function! Moreover the exponential *solves* the heat equation

$$\frac{\partial}{\partial t}\exp(t\mathfrak{D}^2)(f) = \frac{\partial^2}{\partial x^2}\exp(t\mathfrak{D}^2)(f)$$

such that the *initial value* is satisfied

$$\exp(t\mathfrak{D}^2)\Big|_{t=0} = \mathrm{Id}$$

and the *boundary conditions* are satisfied

$$\exp(t\mathfrak{D}^2)(f)(0) = \exp(t\mathfrak{D}^2)(f)(1)$$

$\sum_{n\geq 1} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$ has high regularity for (a_n) with low regularity

Proposition: Let $(a_n)_{n\geq 1}$ be a **bounded** sequence. For every $t > 0$ the series

$$u(t,x) := \sum_{n\geq 1} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

converges uniformly to a **smooth** function

$$u(t,-) \in \mathcal{C}^\infty[0,1]$$

Moreover,

- $$t > 0 \implies u(t,0) = 0 = u(t,1)$$
- $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ on } (t,x) \in (0,\infty) \times [0,1]$$

$u(t,-) \to 0$ as $t \to \infty$

- Each term in $u(t,x)$ is bounded by

$$\left|a_n e^{-(n\pi)^2 t} \sin(n\pi x)\right| \leq (\sup_n |a_n|) e^{-(n\pi)^2 t}$$

- As

$$\sum_{n\geq 1} (\sup_n |a_n|) e^{-(n\pi)^2 t} = (\sup_n |a_n|) \sum_{n\geq 1} (e^{-\pi^2 t})^{n^2} \leq (\sup_n |a_n|) \sum_{n\geq 1} (e^{-\pi^2 t})^n < \infty$$

by

(Weierstrass M-test) Let for $k \in \mathbb{N}$, $f_k : U \to \mathbb{R}$ be a sequence of functions such that

$$x \in X, k \in \mathbb{N} \implies |f_k(x)| \leq M_k$$

and

$$\sum_{k\geq 0} M_k < \infty$$

Then the series

$$\sum_{k\geq 0} f_k$$

converges absolutely and uniformly on U.

we conclude $u(t,x)$ converges uniformly on $x \in [0,1]$ for every $t > 0$.

- $$\left|\sum_{n=N}^{M} a_n e^{-(n\pi)^2 t} \sin(n\pi x)\right| \leq \sum_{n=N}^{M} |a_n| e^{-(n\pi)^2 t} |\sin(n\pi x)|$$
$$\leq \sum_{n=N}^{M} |a_n|$$

- Formally

$$\frac{\partial u}{\partial t} = \sum_{n\geq 1} (-n^2\pi^2) a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

  - On $t \in (0, \infty)$

- Formally

$$\frac{\partial^{2k} u}{\partial x^{2k}} = \sum_{n\geq 1} (-1)^k (n^2\pi^2) a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

[1]

[2]

representation of the Heisenberg algebra

$$\left[\hat{x}, \frac{\mathrm{d}}{\mathrm{d}x}\right]\psi = x\frac{\mathrm{d}}{\mathrm{d}x}\psi - \frac{\mathrm{d}}{\mathrm{d}x}(x\psi) = -\psi$$

Thus

$$\left[\hat{x}, -i\frac{\mathrm{d}}{\mathrm{d}x}\right] = i\mathsf{Id}$$

$$\begin{aligned}\langle I, X, Y \mid [X, Y] = I\rangle &\to \mathrm{End}(L^2[0,1])\\ I &\mapsto (-i)\mathsf{Id}\\ X &\mapsto (-i)\hat{x}\\ Y &\mapsto (-i)\hat{p} := -\mathfrak{D} = -\frac{\mathrm{d}}{\mathrm{d}x}\end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Continuous functions on $\mathbb{R}^d$
      - $\mathcal{C}_0[0,1]$

And it has 5 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Continuous functions on $\mathbb{R}^d$
      - $\mathcal{C}_0[0,1]$
      - Continuous functions on $\mathbb{R}$ with infinite limit are uniformly continuous
      - Intermedia value property of continuous functions on intervals
      - $\mathcal{C}(U, X)$
      - $\mathcal{C}[0,1]$

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1. ocw.mit.edu/courses/18-303-linear-partial-differential-equations-fall-2006/d11b374a85c3fde55ec971fe587f8a50_heateqni.pdf#page=42.00 ↩
2. Ronald B. Guenther _ John W. Lee - Partial Differential Equations of Mathematical Physics and Integral Equations-Prentice Hall (1988), p.163 ↩