Invariant Riemannian metrics on homogeneous manifolds

Let $G \curvearrowright X$ be a transitive and smooth Lie group action. Then X admits a G-invariant Riemannian metric $\iff$ for some (hence all) $x \in X$ the image

$$G_x \to GL(T_xX)$$

has compact closure $\iff$ the image is conjugate to a subgroup of $O(T_xX)$.

Question

Can the image can have not closed image (but pre-compact)?

invariant inner product on one tangent space

Lemma 3.13. *Suppose V is a finite-dimensional real vector space and H is a subgroup of* $\mathrm{GL}(V)$. *There exists an H-invariant inner product on V if and only if H has compact closure in* $\mathrm{GL}(V)$.

Proof. Assume first that there exists an H-invariant inner product $\langle \cdot, \cdot \rangle$ on V. This implies that H is contained in the subgroup $\mathrm{O}(V) \subseteq \mathrm{GL}(V)$ consisting of linear isomorphisms of V that are orthogonal with respect to this inner product. Choosing an orthonormal basis of V yields a Lie group isomorphism between $\mathrm{O}(V)$ and $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ (where $n = \dim V$), so $\mathrm{O}(V)$ is compact; and the closure of H is a closed subset of this compact group, and thus is itself compact.

pushing an invariant inner product everywhere on the manifold

Theorem 3.17 (Existence of Invariant Metrics on Homogeneous Spaces). *Suppose G is a Lie group and M is a homogeneous G-space. Let p_0 be a point in M, and let $I_{p_0}\colon G_{p_0} \to \mathrm{GL}\left(T_{p_0}M\right)$ denote the isotropy representation at p_0. There exists a G-invariant Riemannian metric on M if and only if $I_{p_0}\left(G_{p_0}\right)$ has compact closure in $\mathrm{GL}\left(T_{p_0}M\right)$.*

Proof. Assume first that g is a G-invariant metric on M. Then the inner product g_{p_0} on $T_{p_0}M$ is invariant under the isotropy representation, so it follows from Lemma 3.13 that $I_{p_0}(G_{p_0})$ has compact closure in $\mathrm{GL}\left(T_{p_0}M\right)$.

Conversely, assume that $I_{p_0}\left(G_{p_0}\right)$ has compact closure in $\mathrm{GL}\left(T_{p_0}M\right)$. Lemma 3.13 shows that there is an inner product g_{p_0} on $T_{p_0}(M)$ that is invariant under the isotropy representation. For arbitrary $p \in M$, we define an inner product g_p on T_pM by choosing an element $\varphi \in G$ such that $\varphi(p) = p_0$ and setting

$$g_p = \left(d\varphi_p\right)^* g_{p_0}.$$

If φ_1, φ_2 are any two such elements of G, then $\varphi_1 = h\varphi_2$ with $h = \varphi_1\varphi_2^{-1} \in G_{p_0}$, so

$$\left(d\varphi_1|_p\right)^* g_{p_0} = \left(d(h\varphi_2)_p\right)^* g_{p_0} = \left(d\varphi_2|_p\right)^* \left(dh_{p_0}\right)^* g_{p_0} = \left(d\varphi_2|_p\right)^* g_{p_0},$$

showing that g is well defined as a rough tensor field on M. An easy computation shows that g is G-invariant, so it remains only to show that it is smooth.

The map $\pi\colon G \to M$ given by $\pi(\psi) = \psi \cdot p_0$ is a smooth surjection because the action is smooth and transitive. Given $\varphi \in G$, if we let $\theta_\varphi\colon M \to M$ denote the map $p \mapsto \varphi \cdot p$ and $L_\varphi\colon G \to G$ the left translation by φ, then the map π satisfies

$$\pi \circ L_\varphi(\psi) = (\varphi\psi) \cdot p_0 = \varphi \cdot (\psi \cdot p_0) = \theta_\varphi \circ \pi(\psi), \tag{3.17}$$

so it is equivariant with respect to these two actions. Thus it is a submersion by the equivariant rank theorem (Thm. C.14).

Define a rough 2-tensor field τ on G by $\tau = \pi^* g$. (It will typically not be positive definite, because $\tau_e(v,w) = 0$ if either v or w is tangent to the isotropy group G_{p_0} and thus in the kernel of $d\pi_e$.) For all $\varphi \in G$, (3.17) implies

$$L_\varphi^*\tau = L_\varphi^*\pi^*g = \left(\pi \circ L_\varphi\right)^* g = \left(\theta_\varphi \circ \pi\right)^* g = \pi^*\theta_\varphi^* g = \pi^* g = \tau,$$

where the next-to-last equality follows from the G-invariance of g. Thus τ is a left-invariant tensor field on G. Every basis $(X_1, \dots, X_n)$ for the Lie algebra of G forms a smooth global left-invariant frame for G, and with respect to such a frame the components $\tau(X_i, X_j)$ are constant; thus τ is a smooth tensor field on G.

For each $p \in M$, the fact that π is a surjective smooth submersion implies that there exist a neighborhood U of p and a smooth local section $\sigma\colon U \to G$ (Thm. A.17). Then

$$g|_U = (\pi \circ \sigma)^* g = \sigma^*\pi^* g = \sigma^*\tau,$$

showing that g is smooth on U. Since this holds in a neighborhood of each point, g is smooth. □

6.58 The following result adds to the condition for the existence of an invariant metric on a homogeneous space from 3.4*d*.

Proposition The following conditions are equivalent for a homogeneous space $M = G/H$ (set $p = \not{p}(e) \in M$):

(i) G/H admits a G-invariant Riemannian metric, that is, the structure of a Riemannian homogeneous space.

(ii) M_p admits an inner product invariant under the subgroup λH of $GL(M_p)$, where λ is the linear isotropy representation of H on M_p.

(iii) $\mathfrak{g}/\mathfrak{h}$ admits an Ad_H-invariant inner product.

(iv) The closure of the subgroup λH of $GL(M_p)$ is compact.

(v) The closure of the subgroup Ad_H of $GL(\mathfrak{g}/\mathfrak{h})$ is compact.

Proof The equivalence of (i) and (ii) was proved in 3.4*d*.

The adjoint action of H on $\mathfrak{g}$ is the restriction to H of the usual adjoint action of G on $\mathfrak{g}$; since $\mathrm{Ad}_h X \in \mathfrak{h}$ for $h \in H$ and $X \in \mathfrak{h}$, this induces an "adjoint" action of H on $\mathfrak{g}/\mathfrak{h}$. The representations Ad of H on $\mathfrak{g}/\mathfrak{h}$ and λ of H on M_p are equivalent under the isomorphism $\not{p}_\#: GL(\mathfrak{g}/\mathfrak{h}) \to GL(M_p)$ induced by the projection map $\not{p}$ from G to M, for since $\not{p}(hg) = \not{p}(hgh^{-1})$ for $h \in H$ and $g \in G$, $\lambda(h)\not{p}_*(X + \mathfrak{h}) = \not{p}_* \mathrm{Ad}_h(X + \mathfrak{h}) = (\not{p}_\# \mathrm{Ad}_h)\not{p}_*(X + \mathfrak{h})$ for all $h \in H$ and $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. This implies the equivalence of (ii) and (iii), and of (iv) and (v).

The equivalence of (ii) and (iv) follows by the argument in 6.8.