Isometries of a semi-Riemannian manifold

Proposition:

- Let $(M_1, g_1), (M_2, g_2)$ be semi-Riemannian manifolds with M_1 **connected**. Suppose

$$\varphi, \psi : M_1 \to M_2$$

  are local isometries such that for some $p \in M_1$ we have $\varphi(p) = \psi(p)$ and $\mathfrak{D}_p\varphi = \mathfrak{D}_p\psi$. Then $\varphi \equiv \psi$.
- Let (M, g) be a **connected** Riemannian n-manifold and

$$G \curvearrowright (M, g)$$

  be a smooth, **faithful and isometric** action.
  - For any $p \in M$, the orbit map

$$G/G_p \to M$$

    is an immersion, so $\dim G - \dim G_p = \dim M$ and the derivative/isotopy action

$$G_p \to O(T_pM)$$

    is injective.
  - This implies $\dim G \leq n(n+1)/2$.

Isometry group of a Riemannian manifold

(Myers-Steenrod) Let (M, g) be a Riemannian manifold. Then $\mathrm{Isom}(M, g)$ with the compact-open topology

> Definition. **Compact-open topology on Y^X**
>
> **Let X be a topological space and (Y, d) be a metric space. Let $K \subseteq X$ be compact and $\epsilon > 0$ and consider the subsets**
>
> $$B_\epsilon(f, K) := \left\{ g \in Y^X \middle| \sup_{x \in K} d(f(x), g(x)) < \epsilon \right\}$$
>
> **The topology on Y^X generated by these subsets is the compact-open topology $\mathfrak{T}_{\text{cpt}}$.**

is a Lie group such that the action

$$\mathrm{Isom}(M, g) \curvearrowright M$$

is smooth and proper.

If M is connected, we have

$$\dim \operatorname{Isom}(M,g) \leq \frac{1}{2}\dim M(\dim M+1)$$

with equality $\iff$ M is isometric to $\mathbb{R}^n$ or S^n or $\mathbb{R}\boldsymbol{P}^n$ or $\mathbb{R}\boldsymbol{H}^n$ upto scaling.
If (M,g) is complete and connected?, the action of $G \leq \operatorname{Isom}(M,g)$ is proper $\iff$ G is closed in $\operatorname{Isom}(M,g)$.

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Let $f_n \in \operatorname{Isom}(M,g)$ be such that

$$f_n \xrightarrow{\text{pointwise}, n\to\infty} f$$

- Let $x_0 \in M, \epsilon > 0$. Let $U_\epsilon := B_\epsilon(x_0)$. Then

  $$\forall x \in U_\epsilon, n \in \mathbb{N}, d(f_n(x), f(x_0)) = d(x, x_0) < \epsilon$$

  showing the sequence is equicontinuous

  Definition. Equicontinuous families in $\operatorname{EndTop}(X,Y)$

  Let X be a topological space, (Y,d) be a metric space, and $\mathcal{F} \subseteq \operatorname{EndTop}(X,Y)$. Then

  - $\mathcal{F}$ is **equicontinuous at $x_0 \in X$** if $\forall \epsilon > 0$ there exists an open nb $x_0 \in U \subseteq X$ such that

    $$\forall x \in U, \forall f \in \mathcal{F} \;\; d(f(x), f(x_0)) < \epsilon \iff \forall z \in \mathcal{F}(U), d(z, f(x_0)) < \epsilon$$

  - $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at all points $x \in X$.

- For every x_0, the sequence converges

  $$\{f_n(x_0)\}$$

  so it is bounded.

- Thus by

  (sequential Arzela-Ascoli theorem on σ-compact and locally compact spaces onto metric spaces) Let X be a σ-compact and locally compact topological space and (Y,d) be a metric space. Then a equicontinuous and pointwise bounded subset

  $$\mathcal{F} \subseteq (\operatorname{EndTop}(X,Y), \mathfrak{T}_{\text{cpt}})$$

is **sequentially pre-compact** (any sequence in $\mathcal{F}$ always has a subsequence in $\mathcal{F}$ that converges uniformly on compact subsets of X continuous map $X \to Y$).

f_n converges to f uniformly on compact subsets.

- Now

$$d(x,y) = d(f_n(x), f_n(y)) \xrightarrow{n\to\infty} d(f(x), f(y))$$

thus $f \in \mathrm{Isom}(M,g)$.

Let $x_n \xrightarrow{n\to\infty} x$ and $g_n(x_n) \xrightarrow{n\to\infty} y$.

- ...

Corollaries: Let (M,g) be a Riemannian manifold and $\mathrm{Isom}(M,g)$ be its isometry group and $H \leq \mathrm{Isom}(M,g)$ be a *closed* subgroup.

- For each $x \in M$, the stabilizer/isotopy group $\mathrm{Isom}_x(M,g)$ is **compact**
  - thus

$$\mathrm{Isom}_x(M,g) \to O(T_xM)$$

    is an *embedding*
- For $x \in M$, the H-orbit $H\{x\}$ of x is a *properly embedded submanifold*, as in, the orbit map

$$H/_{H_x} \to H\{x\}$$

  is a **proper embedding**, where

$$H_x = \{h \in H | hx = x\} = \underbrace{\mathrm{Isom}_x(M,g)}_{\text{compact}} \cap H$$

  $\implies$ H_x is **compact**.
  - Thus if an orbit $H\{x\}$ is compact, then H is compact.
  - If $Hx = x \iff H_x = H \implies H$ is compact.
  - In particular if the isometry group $\mathrm{Isom}(M,g)$ has a fixed point, then $\mathrm{Isom}(M,g)$ must be compact.
- If M is compact then $\mathrm{Isom}(M,g)$ is also compact.
  - The converse is in general not true?
  - However, if $\mathrm{Isom}(M,g)$ acts transitively and is compact, then M is also compact.

Closed subgroups of $\mathrm{Isom}(M,g)$

	not proper action/not closed subgroup of $\mathrm{Isom}(M,g)$	proper action/closed subgroup
each stabilizer is compact/finite	dense subgroup of $T^2 \curvearrowright T^2$	any closed subgroup
transitive, with compact stabilizer	possible!	...

Closed transitive subgroups of $\mathrm{Isom}(M,g)$

Which groups appear as $\mathrm{Isom}(M,g)$?

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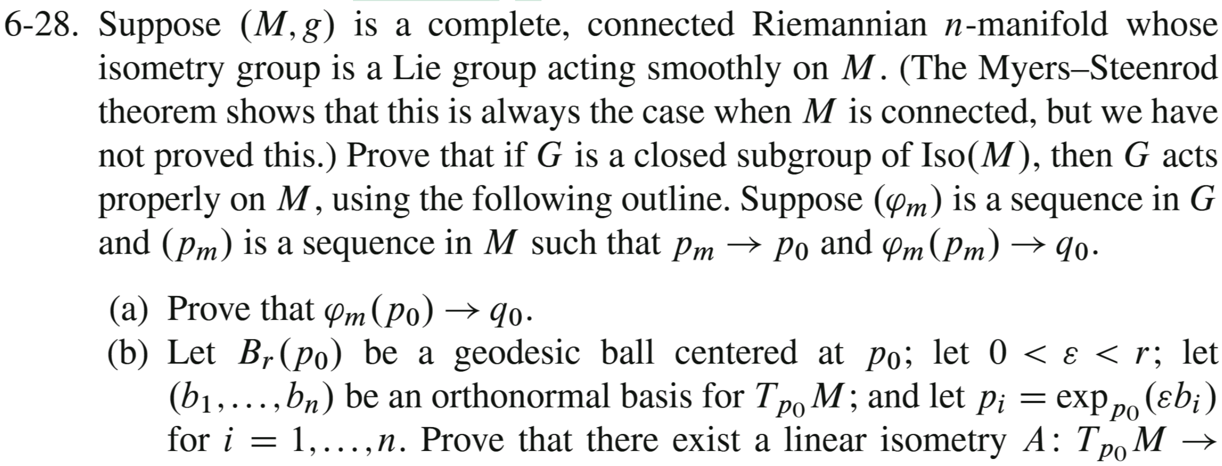
1. Tammo tom Dieck - Transformation Groups-De Gruyter (1987) ↩
2. Marcos M Alexandrino; Renato Ghini Bettiol - Lie Groups and Geometric Aspects of Isometric Actions (2015), page 77 ↩
3. PROPER ISOMETRIC ACTIONS 0811.0547v1.pdf ↩
4. 

6-28. Suppose (M, g) is a complete, connected Riemannian n-manifold whose isometry group is a Lie group acting smoothly on M. (The Myers–Steenrod theorem shows that this is always the case when M is connected, but we have not proved this.) Prove that if G is a closed subgroup of $\mathrm{Iso}(M)$, then G acts properly on M, using the following outline. Suppose (φ_m) is a sequence in G and (p_m) is a sequence in M such that $p_m \to p_0$ and $\varphi_m(p_m) \to q_0$.

(a) Prove that $\varphi_m(p_0) \to q_0$.
(b) Let $B_r(p_0)$ be a geodesic ball centered at p_0; let $0 < \varepsilon < r$; let $(b_1, \dots, b_n)$ be an orthonormal basis for $T_{p_0}M$; and let $p_i = \exp_{p_0}(\varepsilon b_i)$ for $i = 1, \dots, n$. Prove that there exist a linear isometry $A\colon T_{p_0}M \to$

↩
5. Peter Petersen - Riemannian Geometry (2016), p.227 ↩
6. dg.differential geometry - Can every Lie group be realized as the full isometry group of a Riemannian manifold? - MathOverflow ↩