Laplacian on a Riemannian manifold

Definition. Laplacian on a Riemannian manifold

Let (M, g) be an oriented Riemannian manifold with/without boundary. Then the **Laplacian** or the **Laplace-Beltrami operator** on (M, g) is the operator

$$\begin{aligned} \Delta_{\mathrm{g}} : \mathcal{C}^{\infty}(M) &\to \mathcal{C}^{\infty}(M) \\ u &\mapsto \Delta_{\mathrm{g}} u := \mathrm{div}_{\mathrm{g}}(\mathrm{grad}_{\mathrm{g}}(u)) = \mathrm{d}_{\mathrm{g}}^{\dagger} \mathrm{d} u \end{aligned}$$

Proposition:

$$\Delta_{\mathrm{g}}(fh) = f(\Delta_{\mathrm{g}} h) + 2 \left\langle \mathrm{grad}_{\mathrm{g}}(f), \mathrm{grad}_{\mathrm{g}}(h) \right\rangle + h(\Delta_{\mathrm{g}} f)$$

Proposition: In local coordinates,

$$\Delta u = \frac{1}{\sqrt{\det \mathrm{g}_{ij}}} \partial_k \left(\sqrt{\det \mathrm{g}_{ij}}\, \mathrm{g}^{kl} \partial_l u \right)$$