Discrete subgroup of isometry group of a simply connected Riemannian manifold of negative curvature

Definition. Limit set and exponent of a discrete subgroup of isometry group of a simply connected Riemannian manifold of negative curvature

Let X be a simply connected Riemannian manifold of negative curvature with visual boundary ∂X, compactification $\overline{X}$ and $\Gamma \leq \mathrm{Isom}(X)$ be a discrete subgroup. Then for some $x \in X$

- the **limit set** of Γ is

$$\Lambda(\Gamma) := \overline{\Gamma\{x\}} \cap \partial X \subseteq \overline{X}$$

- the **exponent** of Γ is

$$\begin{aligned}\delta_\Gamma &:= \inf\left\{ s \in \mathbb{R} \middle| \sum_{\gamma\in\Gamma} \exp(-s d_X(x,\gamma x)) \right\} \\ &= \limsup_{r\to\infty} \frac{1}{r} \log |\{\gamma \in \Gamma | d(x,\gamma x) \leq r\}|\end{aligned}$$

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Assume the volume entropy of X is finite $\delta_X < \infty$. Then

$$\delta_\Gamma \leq \delta_X$$

If Γ is cocompact then $\delta_\Gamma = \delta_X$ and $\Lambda(\Gamma) = \partial X$.

As Γ is discrete, there exists $s > 0$ such that for all $\gamma \in \Gamma$

$$B_s(\gamma x) \cap B_s(x) \neq \emptyset \implies \gamma \in \Gamma_x$$

- For $r > 0$ we have

$$|\{\gamma \in \Gamma | d(x, \gamma x) \leq r\}| \mu(B_s(x)) \leq |\Gamma_x| B_{r+s}(x)$$

- Let Γ be a cocompact discrete subgroup of $\mathrm{Isom}(X)$.
  - There exists $s > 0$ such that

$$X = \bigcup_{\gamma \in \Gamma} B(\gamma x, s)$$

  - Thus for $r \geq 0$ we have

$$B_r(x) \subset \bigcup_{\substack{\gamma \in \Gamma \\ d(x, \gamma x) \leq r+s}} B_s(\gamma x)$$

Let Γ be a non-elementary discrete group of isometries of X. Then there exists a Γ-conformal density of dimension δ_Γ with support $\Lambda(\Gamma)$.

Let $x \in X$ and suppose $\sum_{\gamma \in \Gamma} \exp(-\delta_\Gamma d(x, \gamma x)) = \infty$.

- Then for

$$\begin{aligned} \Phi : (\delta_\Gamma, \infty) &\to X \\ s &\mapsto \sum_{\gamma \in \Gamma} \exp(-s d(x, \gamma x)) \end{aligned}$$

  we have

$$\nu_s := \frac{1}{\Phi(s)} \sum_{\gamma \in \Gamma} \exp(-s d(x, \gamma x)) \delta_{\gamma x}$$

  a probability measure on $\overline{X}$.
- For a sequence $s_n \xrightarrow{n \to \infty} \delta_\Gamma$ we have a sequence

$$\nu_{s_{n_k}} \xrightarrow{k \to \infty} \nu$$

  weakly
- ...