Boundary of simply connected Hadamard manifolds

Definition. **Boundary of CAT(0) spaces**

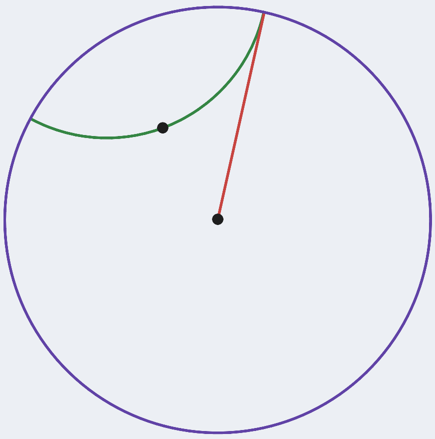
Let X be a complete CAT(0) metric space. Then the set of *asymptotic* equivalence classes of geodesic rays

> **Definition.** **Asymptotic geodesic rays**
>
> Let X be a complete CAT(0) metric space. Two minimal geodesic rays c_1, c_2 are said to be **asymptotic** if there exists a $K > 0$ such that
>
> $$\forall t \geq 0, d(c_1(t), c_2(t)) \leq K$$

is called the **boundary of X**

$$\partial X := \frac{\{c : [0, \infty) \to X \text{ minimal geodesic rays}\}}{\text{asymptotic}}$$

By

> **Proposition:** Let X be a complete CAT(0) space and $c : [0, \infty) \to X$ is a geodesic ray starting at $c(0) = x \in X$. Then for every point $x' \in X$ there **exists an unique** geodesic ray c' which starts at $c'(0) = x'$ and is **asymptotic** to c.
>
> 
>
> This geodesic ray c' shall be denoted $\overline{x'\xi}$ for $\xi := [c(0)] \in \partial X$.

there is a unique bijection

Definition.

$$\begin{aligned} T^{u}_x X &\to \partial X \\ v &\mapsto [\exp_x(tv)] \\ \overline{x\xi}'(0) &\leftarrow \xi \end{aligned}$$

angular metric

Definition. Let $x \in X, y, z \in \overline{X}$ then

$$\angle_x(y,z) := \angle_x(\overline{xy}'(0), \overline{xz}'(0))$$

Proposition: Let $x \in X$

$$\angle_x : \partial X \times \partial X \to [0,\pi]$$

is a metric.

As the angle is a metric on $T^u_x X$, its push to ∂X is also a metric.

cone topology

Definition. Cone topology on $\overline{X}$

Let X be a simply connected Hadamard (CAT(0)) manifold. Let $x \in X, z \in \overline{X}$. For $\epsilon > 0$ then the **cone** at x is

$$c_x(z,\epsilon) := \left\{ y \in \overline{X} \middle| y \neq x, \angle_x(z,y) < \epsilon \right\}$$

The **cone topology** on $\overline{X}$ is

$$\mathfrak{T}_{\overline{X}} := \left\langle \{B_\epsilon(x) | x \in X, \epsilon > 0\} \cup \left\{ c_x(z,\epsilon) \middle| x \in X, z \in \overline{X} \right\} \right\rangle$$

Proposition: The cone topology on X is Hausdorff.

Proposition: Fix $x \in X$, $\{q_i\}$ be a sequence in X, and $\xi \in \partial X$.
Then $q_i \xrightarrow{i\to\infty, \mathfrak{T}_{\overline{X}}} \xi \iff$

$$d(x,q_i) \xrightarrow{i\to\infty} \infty, \measuredangle_x(z,q_i) \xrightarrow{i\to\infty} 0$$

($\implies$) Let $q_i \xrightarrow{i\to\infty, \mathfrak{T}_{\overline{X}}} x$

- Assume $d(x,z_i) \leq R$ (bounded)
  - Then there is a sub-sequence

    $$\exists i_k, r_0 \in \mathbb{R} : \lim_{k\to\infty} d(x, q_{i_k}) = r_0$$

  - But by continuity of $d : X \times X \to \mathbb{R}$ we have

    $$d(x, \lim_{k\to\infty} q_{i_k}) = r_0$$

    which implies $d(x,z) < \infty \implies z \in X$.
  - Thus, a **contradiction**.

Proposition: (Boundary of Hadamard manifolds) Fix $x \in X$ then

Definition.

$$\begin{aligned} T^u_x X &\to \partial X \\ v &\mapsto [\exp_x(tv)] \\ \overline{x\xi}'(0) &\leftarrow \xi \end{aligned}$$

is a homeomorphism.

boundary via horofunctions

Proposition: Consider the embedding

$$X \hookrightarrow \frac{\mathcal{C}(X)}{\mathbb{R}}$$

Then let $h \in \mathcal{C}(X)$

1. h is a Busemann function
2. h is a horofunction
3. h satisfies the following
   - h is convex
   - h is 1-Lipchitz
   - for $x \in X, r > 0$ then

$$\exists x_1, x_2 \in \partial B_r(x) : |h(x_1) - h(x_2)| = 2r$$

4. h is a $\mathcal{C}^1$ convex function with $\left\|\mathrm{grad}_g(h)\right\| = 1$

Proposition?: Let c be any curve, and

$$\mathcal{E}_x := \overline{zc(s)}'(0)$$

Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} d(c(s), z) = \langle c'(0), \mathcal{E}_x \rangle$$

(4 $\implies$ 1) Let $h \in \mathcal{C}^1(X)$ such that $\left\|\mathrm{grad}_g(h)\right\|_g = 1$.

- For $x \in X$ let c_x be the integral curve

$$c_x'(t) = \mathrm{grad}(h)_{c_x(t)}, c_x(0) = x$$

- This means

$$h(c_x(t)) = t + h(x)$$

- c_x is a geodesic

- $c_x(-\infty) = c_y(\infty)$

- By

  Proposition: Fix $x \in X$, $\{q_i\}$ be a sequence in X, and $\xi \in \partial X$.
  Then $q_i \xrightarrow{i \to \infty, \mathfrak{T}_{\overline{X}}} \xi \iff$

  $$d(x, q_i) \xrightarrow{i\to\infty} \infty, \measuredangle_x(z, q_i) \xrightarrow{i \to \infty} 0$$

  we have

$$-\gamma_t'(0) \xrightarrow{t\to\infty} c_y'(0)$$

- Thus

$$\mathrm{grad}(B_{c_x})_y = c_y'(0) = \mathrm{grad}(h)_y$$

Proposition: Let B_{c_1} and B_{c_2} be Busemann functions. Then

$$\begin{aligned}\overline{B_{c_1}} = \overline{B_{c_2}} &\iff \exists x_0 \in X : \mathrm{grad}(B_{c_1})_{x_0} = \mathrm{grad}(B)_{x_0} \\ &\iff c_1(\infty) = c_2(\infty)\end{aligned}$$

Proposition: Let $\{B_i\}_{i\in\mathbb{N}}$ be an open cover of X and

$$f_i : X \to \mathbb{R}$$

be convex $\mathcal{C}^1(X)$ functions...

The map

$$\begin{aligned}\overline{X} &\to \mathrm{Cl}(X) \\ x &\mapsto \overline{d_x} \\ c(\infty) \in \partial X &\mapsto \overline{B_x}\end{aligned}$$

is a homeomorphism.

By

> **Proposition:** Let B_{c_1} and B_{c_2} be Busemann functions. Then
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it is well-defined and injective.

- By

> **Proposition:** Consider the embedding
>
> $$X \hookrightarrow \frac{\mathcal{C}(X)}{\mathbb{R}}$$
>
> Then let $h \in \mathcal{C}(X)$
>
> 1. h is a Busemann function
> 2. h is a horofunction
> 3. h satisfies the following
>   > - h is convex
>   > - h is 1-Lipchitz
>   > - for $x \in X, r > 0$ then
>   >
>   > $$\exists x_1, x_2 \in \partial B_r(x) : |h(x_1) - h(x_2)| = 2r$$
> 4. h is a $\mathcal{C}^1$ convex function with $\left\|\mathrm{grad}_g(h)\right\| = 1$

it is surjective.

maximal metrics

Definition.

$$\angle(\xi,\eta) := \sup_{x\in X} \angle_x(\xi,\eta) : \partial X \times \partial X \to [0,\pi]$$

Proposition:

$$\angle(X) := (\partial X, \angle)$$

is a metric space.

Proposition: Let $\xi,\eta \in \partial X$. Then

$$\lim_{t\to\infty} \angle_{\overline{y\xi}(t)}(\xi,\eta) = \angle(\xi,\eta)$$

4.3 Lemma. Let $z,w \in X(\infty)$, $x \in X$ and $c_i\colon [0,\infty) \longrightarrow X$ be unit speed rays with $c_i(0) = x$, $i=1,2$ with $c_1(\infty) = z$ and $c_2(\infty) = w$. Set $\alpha_t := \measuredangle_{c_1(t)}(x,c_2(t))$, $\beta_t := \measuredangle_{c_2(t)}(x,c_1(t))$. Then $\measuredangle(z,w) = \lim_{t\to\infty}(\pi - \alpha_t - \beta_t)$.

Proposition: Let $\xi,\eta \in \partial X$ and $x \in X$. Let c_1, c_2 be minimal geodesic rays from x to ξ,η respectively. Then

$$t \mapsto \frac{1}{t} d(c_1(t), c_2(t))$$

is monotone increasing and bounded by 2 with limit

$$l(\xi,\eta) := \lim_{t\to\infty} \frac{1}{t} d(c_1(t), c_2(t)) = 2\sin\left(\frac{\angle(\xi,\eta)}{2}\right)$$

For $x \in X$ let c_2, c_2 be the minimal geodesic rays from x to ξ,η and let

$$f(t) := \frac{1}{t} d(c_1(t), c_2(t))$$

-

Definition. Definition

- $$\angle(X)$$

  is the **maximal angular boundary**
- $$l(X)$$

  is the **maximal visual boundary**
- $$\mathrm{Td}(X) := \mathrm{length}(\angle X) = \mathrm{length}(l(X))$$

  is the **Tits boundary**

Proposition: $\angle(X)$ and $l(X)$ are complete metric spaces.

💡 Let ξ_i be a Cauchy sequence in $\angle(X)$.

- Because

$$\angle_x(\xi_n, \xi_m) \leq \angle(\xi_n, \xi_m)$$

  for all $x \in X$.
- ...

Proposition: Let $\xi, \eta \in \mathrm{Td}(X)$. If there is **no** geodesic $c : \mathbb{R} \to X$ such that $c(-\infty) = \xi, c(\infty) = \eta$ then

$$\mathrm{Td}(\xi, \eta) = \angle(\xi, \eta) \leq \pi$$

Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature (1985), p.40

Proposition:

$$\mathrm{length}(\angle X) = \mathrm{length}(l(X))$$

Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature (1985), p.43

global geometry of Tits boundary

Proposition:

- $$\mathrm{Td}(\xi, \eta) > \pi$$

  $\implies$ there is a geodesic $c : \mathbb{R} \to X$ with $c(-\infty) = \xi, c(\infty) = \eta$
- there is a geodesic $c : \mathbb{R} \to X$ with $c(-\infty) = \xi, c(\infty) = \eta \implies$

$$\mathrm{Td}(\xi, \eta) \geq \pi$$

- $$\mathrm{diam}(\mathrm{Td}(X)) \geq \pi$$

  with equality $\iff$ any geodesic in X bounds a flat half plane
- ...

Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature (1985), p.46