Homogeneous and isotropic Riemannian manifolds

Proposition: Let (M, g) be a Riemannian manifold such that for each $p \in M$ for any $X_p, Y_p \in T_pM$, there exists an isometry $f : (M, g) \to (M, g)$ such that

$$\mathfrak{D}_p f(X_p) = Y_p$$

Then (M, g) is **homogeneous** and isotropic.

Let $p, q \in M$ with a minimizing geodesic arc

$$\gamma : [0, 2a] \to M$$

connecting p, q (not necessarily unique). Then consider the velocity at $\gamma(a)$

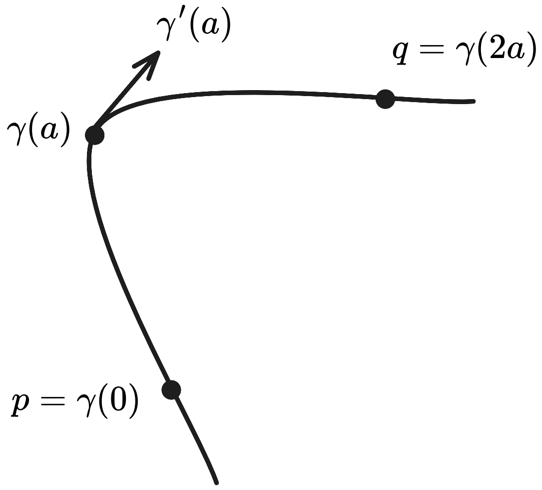
$$\gamma'(a)$$

- By assumption, there is an isometry $f : (M, g) \to (M, g)$ that maps

$$\gamma'(a) \mapsto -\gamma'(a)$$

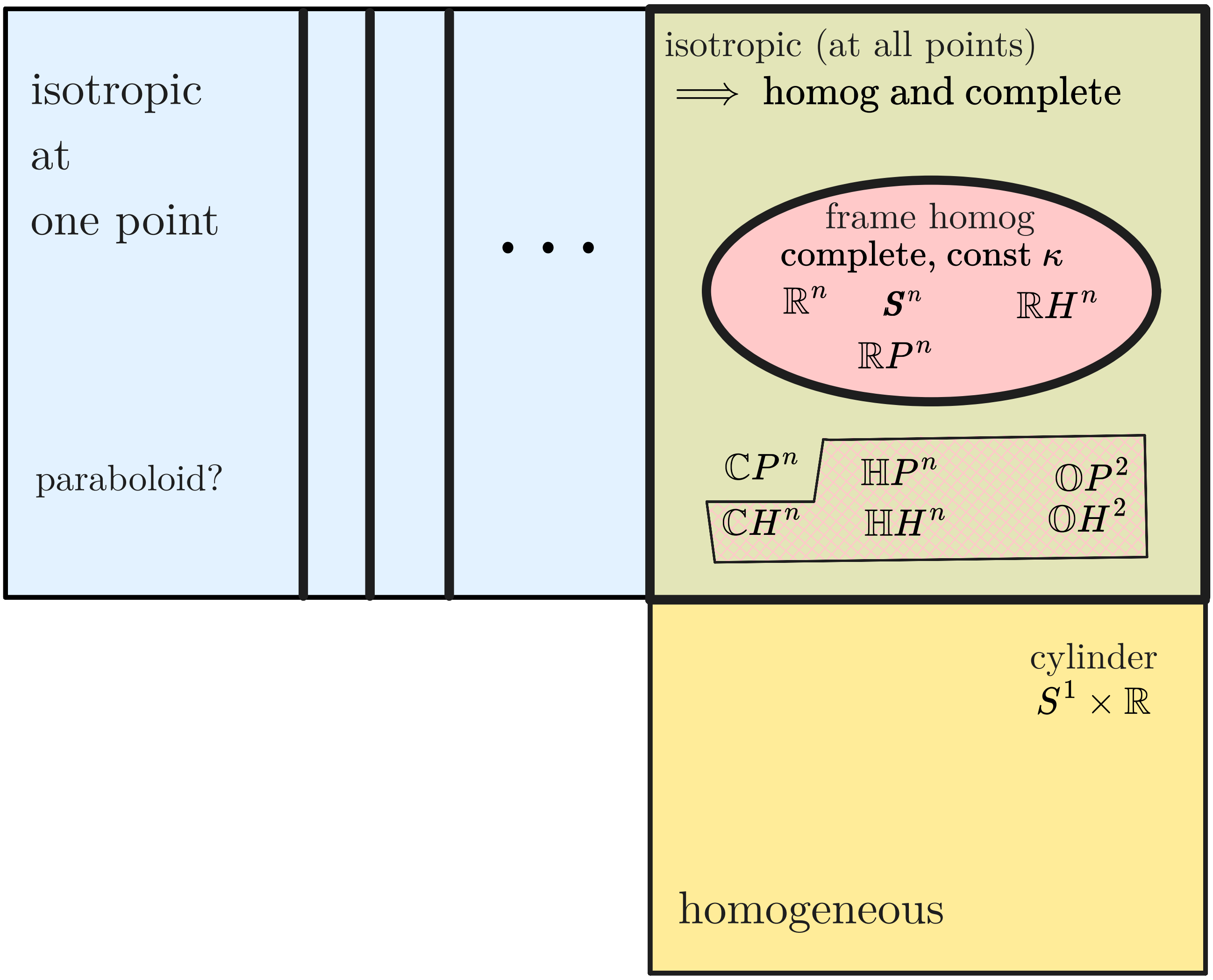
- Then

$$\begin{aligned} f(q) &= f(\exp_{\gamma(a)}(\gamma'(a))) \\ &= \exp_{\gamma(a)}(\mathfrak{D}_a f(\gamma'(a))) \\ &= \exp_{\gamma(a)}(-\gamma'(1)) \\ &= p \end{aligned}$$



- Thus, (M, g) is homogeneous.

connected Riemannian manifolds



(Killing-Hopf) Let (M,g) be a complete, simply connected Riemannian $n \geq 2$-manifold with constant sectional curvature. Then M is isometric to one of

$$\mathbb{R}^n, (S^n, Rg), (\mathbb{R}H^n, Rg)$$

space	$\dim_{\mathbb{R}}$	type	quotient	sectional curvature
$\mathbb{R}^n$	n	flat	$E(n,\mathbb{R})/O(n,\mathbb{R})$	0
S^n	n	compact	$SO(n+1,\mathbb{R})/SO($	+1
$\mathbb{R}P^n$	n	compact	$SO(n+1,\mathbb{R})/O(n,$	+1
$\mathbb{C}P^n$	$2n$	compact	$SU(n+1,\mathbb{C})/U(n,$	
$\mathbb{H}P^n$		compact	$Sp(n+1)/Sp(n)\times$	
$\mathbb{O}P^2$		compact		
$\mathbb{R}H^n$	n	non-compact dual of $S^n, \mathbb{R}P^n$	$SO^+(1,n+1,\mathbb{R})/$	−1

space	$\dim_{\mathbb{R}}$	type	quotient	sectional curvature
$\mathbb{C}\boldsymbol{H}^n$		non-compact dual of $\mathbb{C}\boldsymbol{P}^n$	$SU(1,n+1)/U(n,$	
$\mathbb{H}\boldsymbol{H}^n$		non-compact dual of $\mathbb{H}\boldsymbol{P}^n$	$Sp(1,n+1)/Sp(n)$	
$\mathbb{O}\boldsymbol{H}^2$		non-compact dual of $\mathbb{O}\boldsymbol{P}^2$		

classification

Let M be a connected Riemannian manifold. Then $\mathrm{Isom}(M)$ acts transitively on equidistant pairs of distinct points

$$\iff$$

M Riemannian manifold which is isotropic at all points

$$\iff$$

M is isometric to $\mathbb{R}^n$ or a symmetric space of rank 1

$$\iff$$

M is isometric to one of

space	$\dim_{\mathbb{R}}$	type	quotient	sectional curvature
$\mathbb{R}^n$	n	flat	$E(n,\mathbb{R})/O(n,\mathbb{R}$	0
S^n	n	compact	$SO(n+1,\mathbb{R})/S$	$+1$
$\mathbb{R}\boldsymbol{P}^n$	n	compact	$SO(n+1,\mathbb{R})/C$	$+1$
$\mathbb{C}\boldsymbol{P}^n$	$2n$	compact	$SU(n+1,\mathbb{C})/U$	
$\mathbb{H}\boldsymbol{P}^n$		compact	$Sp(n+1)/Sp(n$	
$\mathbb{O}\boldsymbol{P}^2$		compact		
$\mathbb{R}\boldsymbol{H}^n$	n	non-compact dual of $S^n, \mathbb{R}\boldsymbol{P}^n$	$SO^+(1,n+1,\mathbb{R}$	-1
$\mathbb{C}\boldsymbol{H}^n$		non-compact dual of $\mathbb{C}\boldsymbol{P}^n$	$SU(1,n+1)/U($	
$\mathbb{H}\boldsymbol{H}^n$		non-compact dual of $\mathbb{H}\boldsymbol{P}^n$	$Sp(1,n+1)/Sp$	
$\mathbb{O}\boldsymbol{H}^2$		non-compact dual of $\mathbb{O}\boldsymbol{P}^2$		

upto a constant scaling.

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1. Joseph A. Wolf - Spaces of Constant Curvature (2011, AMS) - 6th ed, p.315 ↩
2. https://math.stackexchange.com/a/4920625 ↩
3. https://math.stackexchange.com/a/4921317 ↩