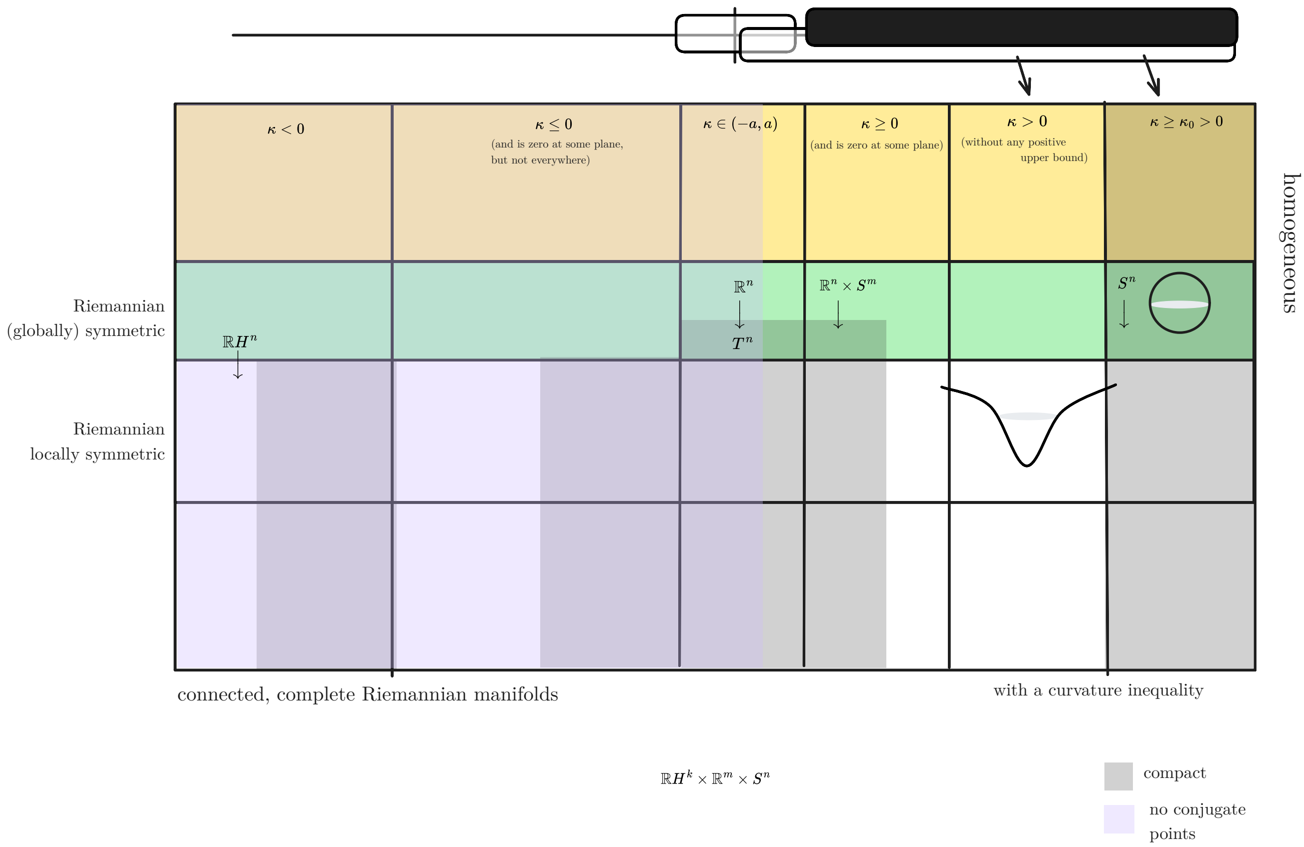
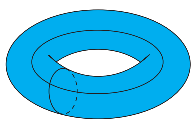
Complete Riemannian manifolds





completeness

(Ambrose) Let

$$\pi : \underbrace{(X,h)}_{\text{complete}} \to (M,g)$$

be a local isometry of connected Riemannian manifolds with (X,h) complete. Then (M,g) is complete and π is a Riemannian covering map.

Warning

If

$$\pi : (X,h) \to \underbrace{(M,g)}_{\text{complete}}$$

is a local isometry, then it **not** guaranteed that π is a covering.

Let

$$\pi : (X, h) \to (M, g)$$

be a Riemannian covering map between connected Riemannian manifolds. Then (M, g) is complete $\iff$ (X, h) is complete.

- One side is from

> **(Ambrose) Let**
>
> $$\pi : \underbrace{(X, h)}_{\text{complete}} \to (M, g)$$
>
> **be a local isometry of connected Riemannian manifolds with (X, h) complete. Then (M, g) is complete and π is a Riemannian covering map.**

- Conversely, if M is complete, lifts of geodesics on M are geodesics on X.

Let (M, g) be a connected Riemannian manifold and $p \in M$. If

$$\exp_p : T_pM \to M$$

is (well-defined and) a local diffeomorphism everywhere, then $\exp_p$ is a smooth covering map.

By

For any $v \in T_pM$ the curve $t \mapsto tv$ is a geodesic since its image is a geodesic in (M, g). Thus

constant curvature

(Killing-Hopf) Let (M, g) be a complete, simply connected Riemannian $n \geq 2$-manifold with constant sectional curvature. Then M is isometric to one of

$$\mathbb{R}^n, (S^n, Rg), (\mathbb{R}H^n, Rg)$$

Corollary 12.5 (Characterization of Constant-Curvature Manifolds). *The complete, connected, n-dimensional Riemannian manifolds of constant sectional curvature are, up to isometry, exactly the Riemannian quotients of the form $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is one of the constant-curvature model spaces $\mathbb{R}^n$, $\mathbb{S}^n(R)$, or $\mathbb{H}^n(R)$, and Γ is a discrete subgroup of $\operatorname{Iso}(\widetilde{M})$ that acts freely on $\widetilde{M}$.*

positive

(Bonnet-Myers) Let (M, g) be a complete connected Riemannian n-manifold and suppose there is a $R > 0$ such that

$$\mathrm{Ric}_g(v,v) \geq \frac{n-1}{R^2}$$

for all unit vectors $v \in T^{\mathsf{u}}(M,g)$. Then M is compact with

$$\mathrm{diam}(M,g) \leq \pi R$$

and $\pi_1(M)$ is finite.