Uniquely geodesic complete Riemannian manifolds

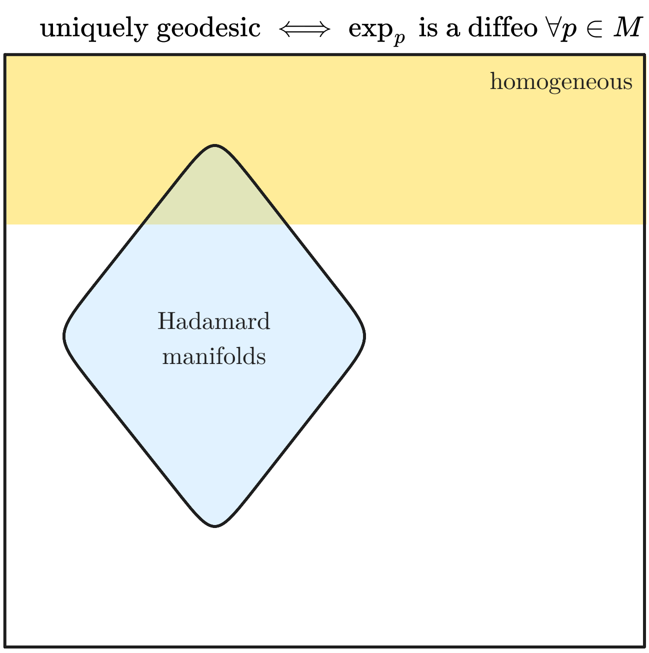
Proposition: Let (M, g) be a complete Riemannian manifold. Then the following are equivalent

1. every geodesic in (M, g) is (globally) **length-minimizing**
2. every unit speed geodesic in (M, g) is a minimal geodesic
3. every pair of points in (M, g) is joined by a **unique** minimal geodesic segment
4. the exponential

$$\exp_p : T_pM \to M$$

is a diffeomorphism for all $p \in M$

[1]



embedded *completely* geodesic submanifolds

Definition. Embedded *completely* geodesic submanifolds

Let $S \subseteq M$ be a connected, *embedded* submanifold of a uniquely geodesic Riemannian manifold (M, g). Then S is called **completely geodesic** if for every g-geodesic $\gamma : \mathbb{R} \to M$, if γ meets S in two points then this implies $\gamma(\mathbb{R}) \subset \iota(S)$.

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2. $$\forall q \in S, \exp_q^{\mathrm{g}}(T_q S) \subseteq \iota(S)$$

In any (hence all) of these cases

- for any $p \in S$, $\exp_p^{(M,\mathrm{g})}(T_p S) = S$
- $d_{\iota^*\mathrm{g}} = \iota^* d_{\mathrm{g}}$
- $(S, \iota^*\mathrm{g})$ is a complete Riemannian manifold

Assume (1). Then

$$d_{\iota^*\mathrm{g}} = \iota^* d_{\mathrm{g}}$$

- Thus a g-geodesic must agree with ι^*g-geodesic.
- Thus (2) follows.

Assume (2). Let $p, q \in S$. Then p, q

- By

Proposition:
Let

$$\iota : (M, \iota^*\mathrm{g}) \hookrightarrow (X, \mathrm{g})$$

be an *embedded* Riemannian submanifold. Then the following is equivalent

- M is totally geodesic

Definition. Totally geodesic submanifolds of semi-Riemannian manifolds

A Riemannian (immersed/embedded) submanifold

$$\iota : (M, \mathrm{g}_M) \hookrightarrow (X, \mathrm{g}_X)$$

of a semi-Riemannian manifold (X, g_X) is **totally geodesic** if for every g_X-geodesic that is tangent to M at some $t_0 \in \mathbb{R}$ stays in M for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$
.

- Every ι^*g-geodesic $\gamma : (a, b) \to M$ is also a g-geodesic $\iota \circ \gamma : (a, b) \to X$
- The second fundamental form of M vanishes identically.

we have ...

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1. https://math.stackexchange.com/a/1530561/1290493 ↩