Riemannian connection

- *Torsion-free*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

- *metric-compactible*

$$X(g(Y, Z)) = g(\nabla_X, Z) + g(Y, \nabla_X Z)$$

$$\begin{aligned} \nabla : \mathrm{Vec}(M) &\to \mathrm{Vec}(M) \otimes \Omega^1(M) =? \boldsymbol{\Gamma}(T^*M \otimes TM) \\ X &\mapsto \nabla_{(-)} X \end{aligned}$$

$$[X, Y] \mapsto \nabla[X, Y] =?$$

expression

We compute

$$X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y))$$

and obtain

$$g(\nabla_X Y, Z) = \begin{matrix} X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ -g(Y, [X, Z]) - g(Z, [Y, X]) + g(X, [Z, Y]) \end{matrix}$$

the covector field of $\nabla_X Y$ is

$$X(g(Y, -)) + Y(g(-, X)) - \mathrm{d}(g(X, Y)) - g(Y, [X, -]) - g(-, [Y, X]) + g(X, [-, Y])$$

in a chart

On a chart (U, x) we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_m \Gamma_{ij}^m \frac{\partial}{\partial x_m}$$

because $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ we have