Curvature of a Riemannian manifold

Definition. Riemann curvature tensor of a Riemannian manifold

The assignment

$$(X, Y, Z) \mapsto ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$$

produces a (1,3)-tensor on M.
The **Riemann curvature tensor** is the (0,4)-tensor

$$R(X, Y, Z, W) := g(([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z, W)$$

The Riemann curvature (0,4)-tensor is actually

$$R \in \Gamma \boldsymbol{Sym}^2 \boldsymbol{\Lambda}^2 TM$$

and it satisfies the two Branchi identities

$$R(W, X, Y, Z) + R(X, Y, W, Z) + R(Y, W, X, Z) = 0$$

Definition. Ricci and scalar tensors

$$\mathrm{Ric}(X, Y) := \mathrm{tr}(Z \mapsto)$$

		R^l_{ijk}
Riemann	$R : \boldsymbol{Sym}^2 \boldsymbol{\Lambda}^2 TM \to \mathbb{R}$	$R_{ijkl} = g_{lm} R^m_{ijk}$
	we may think of it as a *symmetric bilinear form on the 2-wedge* tangent space $\boldsymbol{\Lambda}^2 TM \times \boldsymbol{\Lambda}^2 TM \to \boldsymbol{Sym}^2 \boldsymbol{\Lambda}^2 TM \xrightarrow{R}$ $(X \wedge Y, Z \wedge W) \mapsto R(X, Y, Z, W)$	
	and consider the *quadratic form* of the above bilinear form $\boldsymbol{\Lambda}^2 TM \to \mathbb{R}$ $R(X \wedge Y) \mapsto R(X, Y, X, Y)$	
sectional	$\kappa : \boldsymbol{Gr}^2 TM \to \mathbb{R}$ $\kappa(\mathbb{R}\langle v, w\rangle) := \frac{R(v, w, w, v)}{\lVert v \wedge w \rVert^2_{\mathrm{g}}}$	
	$(X, Y) \mapsto \mathrm{tr}(Z \mapsto R(X, Z)Y)$	$\mathrm{g}^{pq} R_{ipjq}$

Ricci	$\mathrm{Ric} : T^{\mathsf{u}}(M, g) \to \mathbb{R}$	
scalar	$\mathrm{sc} := \mathrm{tr}(\mathrm{Ric}) : M \to \mathbb{R}$	$\kappa = R^i_i = \mathrm{g}^{ij} R_{ij}$
total scalar curvature	$\int_M \mathrm{sc}_{\mathrm{g}} \lambda_{\mathrm{g}}$	