Totally geodesic submanifolds of semi-Riemannian manifolds

Definition. Totally geodesic submanifolds of semi-Riemannian manifolds

A Riemannian (immersed/embedded) submanifold

$$\iota : (M, g_M) \hookrightarrow (X, g_X)$$

of a semi-Riemannian manifold (X, g_X) is **totally geodesic** if for every g_X-geodesic that is tangent to M at some $t_0 \in \mathbb{R}$ stays in M for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Proposition:
Let

$$\iota : (M, \iota^* g) \hookrightarrow (X, g)$$

be an *embedded* Riemannian submanifold. Then the following is equivalent

- M is totally geodesic

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- Every $\iota^* g$-geodesic $\gamma : (a, b) \to M$ is also a g-geodesic $\iota \circ \gamma : (a, b) \to X$
- The second fundamental form of M vanishes identically.

[1]

[2]

Proposition 8.12. *Suppose (M, g) is an embedded Riemannian submanifold of a Riemannian or pseudo-Riemannian manifold $(\widetilde{M}, \widetilde{g})$, The following are equivalent:*

(a) *M is totally geodesic in $\widetilde{M}$.*
(b) *Every g-geodesic in M is also a $\widetilde{g}$-geodesic in $\widetilde{M}$.*
(c) *The second fundamental form of M vanishes identically.*

1. ↩

2. alberto-2024.pdf ↩