Connected Lie groups with non-surjective exponential map

Proposition: There is *no* Riemannian metric on a Lie group G with non-surjective exponential map such that

$$\exp_{i_G} = \exp : \mathrm{Lie}(G) \to G$$

Let there is such a metric, then by

(Hopf-Rinow) Let (M, g) be a connected Riemannian manifold. Then M is geodesically complete

$\Longleftrightarrow$

(M, d_g) is a complete metric space

$\Longleftrightarrow$

$\exp_p : T_pM \to M$ is defined for some $p \in M$

$\Longleftrightarrow$

every closed d-bounded set in M is compact.
If (M, g) satisfies any (hence all) of the above, then each pair of points in M can be joined by a length-minimizing geodesic segment. In particular, for each $p \in M$ the exponential

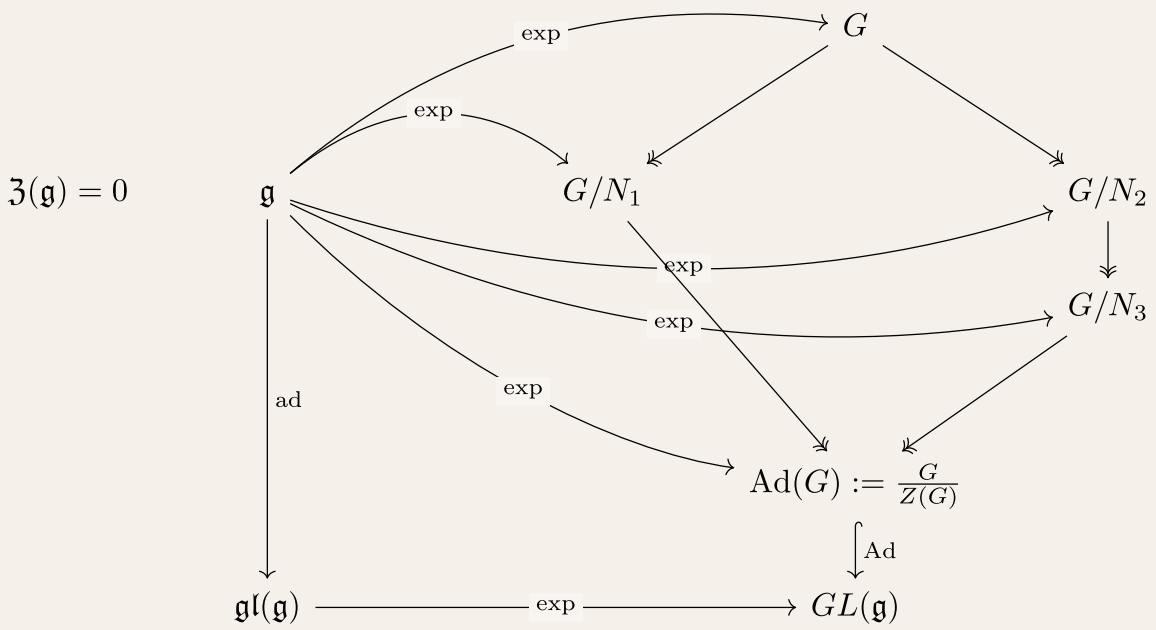
$$\exp_p : T_pM \to M$$

is surjective.

it is complete, hence $\exp = \exp_{i_G}$ must be surjective.

Covering of a *connected Lie group with non-surjective exponential map*

Connected Lie groups with non-surjective exponential map and discrete center



- If exponential is *not* surjective onto $\mathrm{Ad}(G)$ then it it not surjective onto any Lie group with Lie algebra $\mathfrak{g}$.