Almost simple groups

Definition. A Lie group is almost simple if its Lie algebra is *simple*.

By

Let G be a connected semi-simple Lie group. Then G is compact $\iff$ the Killing form on $\mathrm{Lie}(G)$ is negative-definite.

Theorem 2.35. *Let G be a n-dimensional semisimple connected Lie group. Then G is compact if and only if its Killing form B is negative-definite.*

Proof. First, suppose that B is negative-definite. From Corollary 2.33, $-B$ is a bi-invariant metric on G. Hence, the Hopf-Rinow Theorem 2.9 and Theorem 2.30 imply that $(G,-B)$ is a complete Riemannian manifold, whose Ricci curvature satisfies (2.15). It follows from the Bonnet-Myers Theorem 2.19 that G is compact.

Conversely, suppose G is compact. From Proposition 2.24, it admits a bi-invariant metric Q. Hence, using item (i) of Propositions 2.26 and 1.37, it follows that, if $(e_1,\dots,e_n)$ is an orthonormal basis of $\mathfrak{g}$, then

$$\begin{aligned} B(X,X) &= \mathrm{tr}\left(\mathrm{ad}(X)\mathrm{ad}(X)\right) \\ &= \sum_{i=1}^{n} Q(\mathrm{ad}(X)\mathrm{ad}(X)e_i, e_i) \\ &= -\sum_{i=1}^{n} Q(\mathrm{ad}(X)e_i, \mathrm{ad}(X)e_i) \\ &= -\sum_{i=1}^{n} \|\mathrm{ad}(X)e_i\|^2 \leq 0. \end{aligned}$$

Note that, if there exists $X \neq 0$ such that $\|\mathrm{ad}(X)e_i\|^2 = 0$ for all i, then by definition of the Killing form, $B(Y,X) = 0$ for each Y. This would imply that B is degenerate, contradicting the fact that $\mathfrak{g}$ is semisimple. Thus, for each $X \neq 0$, we have $B(X,X) < 0$. Therefore, B is negative-definite. □

The next result follows directly from Corollary 2.33, Remark 2.34 and Theorem 2.35.

every connected Lie group corresponding to a Almost simple group with negative-definite Killing form is **compact**.

Almost simple algebra with negative-definite Killing form, AKA compact Lie $\mathbb{R}$-algebras	compact simple connected Lie groups	Z
$\mathfrak{su}(2,\mathbb{C})$	$SU(2,\mathbb{C}) \to SO(3,\mathbb{R})$	$\mathbb{Z}_2$

And *every* connected Lie group corresponding to a *almost simple algebra* whose Killing form is not negative-definite is **non-compact**.

Almost simple algebra whose Killing form is not negative-definite, AKA non-compact Lie $\mathbb{R}$-algebra	signature of Killing form	non-compact simple connected Lie groups	Z
$\mathfrak{sl}(2,\mathbb{R})$	$(1,2)$	$\widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R}) \quad \cong SO^+(1,2$	$\mathbb{Z}$
$\mathfrak{sl}(n,\mathbb{R})$ for $n \geq 3$			

Classification of simple Lie $\mathbb{R}$-algebras

(Classification of simple Lie $\mathbb{R}$-algebras into complex and non-complex) Let $\mathfrak{g}$ be a simple Lie $\mathbb{R}$-algebra and $\mathfrak{g} \otimes \mathbb{C}$ be its complexification. Then either it is **complex**:

$$\mathfrak{g} \cong_{\mathsf{LieAlg}_{\mathbb{R}}} \mathfrak{s}$$

for some Lie $\mathbb{C}$-algebra $\mathfrak{s}$ (and $\mathfrak{g} \otimes \mathbb{C} \cong_{\mathsf{LieAlg}_{\mathbb{C}}} \mathfrak{s} \oplus \mathfrak{s}$) *or* it is **non-complex**: $\mathfrak{g} \otimes \mathbb{C}$ is simple over $\mathbb{C}$.

	simple Lie $\mathbb{C}$-algebra	compact simple Lie $\mathbb{R}$-algebra (*compact real form*)	non-compact non-complex simple Lie $\mathbb{R}$-algebras (*non-compact real forms*)
$A_n, n \geq 1$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{su}(n+1, \mathbb{C})$	$\mathfrak{sl}(n+1, \mathbb{R})$
			$\mathfrak{su}(p, q)$
			$\mathfrak{sl}(m, \mathbb{H})$
$B_n, n \geq 2$	$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n+1, \mathbb{R})$	$\mathfrak{so}(p, q, \mathbb{R})$
$C_n, n \geq 3$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{usp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R})$
			$\mathfrak{so}(p, 2n-p, \mathbb{R})$
$D_n, n \geq 4$	$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n, \mathbb{R})$	$\mathfrak{so}(p, q, \mathbb{R})$
			$\mathfrak{so}(2n, \mathbb{H})$
E_6			
E_7			
E_8			
F_4			
G_2			

Theorem 6.105 (classification). Up to isomorphism every simple real Lie algebra is in the following list, and everything in the list is a simple real Lie algebra:

(a) the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is complex simple of type A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4, or G_2,

(b) the compact real form of any $\mathfrak{g}$ as in (a),

(c) the classical matrix algebras

$$
\begin{array}{lrl}
\mathfrak{su}(p,q) & \text{with} & p \geq q > 0,\ p+q \geq 2 \\
\mathfrak{so}(p,q) & \text{with} & p > q > 0,\ p+q \text{ odd},\ p+q \geq 5 \\
 & \text{or with} & p \geq q > 0,\ p+q \text{ even},\ p+q \geq 8 \\
\mathfrak{sp}(p,q) & \text{with} & p \geq q > 0,\ p+q \geq 3 \\
\mathfrak{sp}(n,\mathbb{R}) & \text{with} & n \geq 3 \\
\mathfrak{so}^*(2n) & \text{with} & n \geq 4 \\
\mathfrak{sl}(n,\mathbb{R}) & \text{with} & n \geq 3 \\
\mathfrak{sl}(n,\mathbb{H}) & \text{with} & n \geq 2,
\end{array}
$$

(d) the 12 exceptional noncomplex noncompact simple Lie algebras given in Figures 6.2 and 6.3.

The only isomorphism among Lie algebras in the above list is $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$.