Orbits of smooth Lie group actions

Let $\theta : G \curvearrowright M$ be a smooth left Lie group action on a smooth manifold M and $p \in M$. Then the orbit map

$$\begin{aligned}\theta_{(p)} : G &\to M \\ g &\mapsto gp\end{aligned}$$

is smooth and has constant rank, so the isotopy group $G_p = \theta_{(p)}^{-1}(p)$ is a closed subgroup of G. Then

$$\theta_{(p)} : \frac{G}{G_p} \hookrightarrow M$$

is an injective immersion making the image, that is, the orbit $\theta^G(p)$ is an immersed submanifold

$$\iota : (G\{p\}, \theta_{(p)*}\mathfrak{T}_{G/G_p}, \theta_{(p)*}\mathcal{A}_{G/G_p}) \hookrightarrow M$$

Moreover, if the action θ is *proper* then the orbit map $\theta_{(p)}$ is a proper smooth embedding (so in particular the orbit $\theta^G(p)$ is closed/properly embedded) and each stabilizer G_p is compact.

Proposition 7.26 (Properties of the Orbit Map). *Suppose θ is a smooth left action of a Lie group G on a smooth manifold M. For each $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is smooth and has constant rank, so the isotropy group $G_p = \left(\theta^{(p)}\right)^{-1}(p)$ is a properly embedded Lie subgroup of G. If $G_p = \{e\}$, then $\theta^{(p)}$ is an injective smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M.*

Proposition 21.7 (Orbits of Proper Actions). *Suppose θ is a proper smooth action of a Lie group G on a smooth manifold M. For any point $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is a proper map, and thus the orbit $G \cdot p = \theta^{(p)}(G)$ is closed in M. If in addition $G_p = \{e\}$, then $\theta^{(p)}$ is a smooth embedding, and the orbit is a properly embedded submanifold.*

21-17. Suppose a Lie group acts smoothly (but not necessarily properly or freely) on a smooth manifold M. Show that each orbit is an immersed submanifold of M, which is embedded if the action is proper.

	orbit $G\{p\}$	$G_p \leq G$	M/G
(general)	immersed submanifold of M	closed	(bad)
proper	closed submanifold of M, diffeomorphic to G/G_p	compact	Hausdorff

	orbit $G\{p\}$	$G_p \leq G$	M/G
G is compact $\implies$ action is proper	compact submanifold of M, diffeomorphic to G/G_p	compact	Hausdorff
free		trivial	
proper, free		trivial	smooth manifold
transitive	orbit is M, diffeomorphic to G/G_p		point (smooth manifold)

action	orbit $G\{p\}$	
not proper	embedded but not closed	$$\mathbb{R}_{>0} \curvearrowright \mathbb{R}$$ whose orbits other than $\{0\}$ are $$(-\infty, 0), (0, \infty)$$ which are *not closed*
not proper	closed and embedded	
not proper	not embedded, closed	impossible?
not proper	not embedded, not closed	

Bug

If an orbit is closed, then it is an embedded submanifold? [1]

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1. https://mathoverflow.net/a/418306/562926 ↩