Moduli space of quotients

Proposition:

3-21. Let $(\widetilde{M}, \widetilde{g})$ be a simply connected Riemannian manifold, and suppose Γ_1 and Γ_2 are countable subgroups of $\operatorname{Iso}(\widetilde{M}, \widetilde{g})$ acting smoothly, freely, and properly on $\widetilde{M}$ (when endowed with the discrete topology). For $i = 1, 2$, let $M_i = \widetilde{M}/\Gamma_i$, and let g_i be the Riemannian metric on M_i that makes the quotient map $\pi_i : \widetilde{M} \to M_i$ a Riemannian covering (see Prop. 2.32). Prove that the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are isometric if and only if Γ_1 and Γ_2 are conjugate subgroups of $\operatorname{Iso}(\widetilde{M}, \widetilde{g})$.

Definition. Teichmuller space

Let X be simply connected, π_1 be the fundamental group of a covering quotient of X. Then the *space* of all discrete faithful representations of π_1 in $\operatorname{Isom}(X, \mathrm{g})$ is

$$\begin{aligned}\mathrm{DF}(\pi_1, \operatorname{Isom}(X, \mathrm{g})) := \{\rho : \pi_1 \hookrightarrow \operatorname{Isom}(X, \mathrm{g}) | \operatorname{im} \rho \text{ is discrete}\} \\ \subseteq \mathrm{Hom}\mathsf{GrpTop}(\pi_1, \operatorname{Isom}(X, \mathrm{g}))\end{aligned}$$

Then isometry group $\operatorname{Isom}(X, \mathrm{g})$ acts on it by *conjugation*

$$\begin{aligned}\mathrm{DF}(\pi_1, \operatorname{Isom}(X, \mathrm{g})) \curvearrowleft \operatorname{Isom}(X, \mathrm{g}) \\ \rho \xrightarrow{h} h\rho h^{-1}\end{aligned}$$

Then

$$\operatorname{Teich}(\pi_1, \operatorname{Isom}(X, \mathrm{g})) := {}^{\mathrm{DF}(\pi_1, \operatorname{Isom}(X, \mathrm{g}))}\Big/_{\operatorname{Isom}(X, \mathrm{g})}$$

is the **Teichmuller space** for π_1 in $\operatorname{Isom}(X, \mathrm{g})$.

This is the *space* of all *marked* quotients $_{\pi_1}\backslash^{X}$, as in with a specified representation

$$\pi_1 \hookrightarrow \operatorname{Isom}(X, \mathrm{g})$$

(but it could be possible that some $\operatorname{im} \rho$ do not act *freely*?)

However, we could have composed with an automorphism Φ of $\pi_1(X)$

$$\pi_1 \overset{\Phi}{\cong} \pi_1 \overset{\rho}{\hookrightarrow} \operatorname{Isom}(X, \mathrm{g})$$

which *may* be a different element in the *Teichmuller space* but by the proposition

$$_{\rho(\pi_1)}\backslash^{X} \cong {}_{\rho \circ \Phi(\pi_1)}\backslash^{X}$$

so corresponds to an isomorphic quotient.

Thus

Definition. Moduli space

The action

$$\mathrm{AutGrp}(\pi_1) \curvearrowright {}^{\mathrm{DF}(\pi_1, \mathrm{Isom}(X, \mathrm{g}))}/_{\mathrm{Isom}(X, \mathrm{g})}$$

has the quotient space

$$\mathrm{Moduli}(\pi_1, \mathrm{Isom}(X, \mathrm{g})) := {}_{\mathrm{AutGrp}(\pi_1)}\backslash^{\mathrm{DF}(\pi_1, \mathrm{Isom}(X, \mathrm{g}))/_{\mathrm{Isom}(X, \mathrm{g})}}$$

called the **moduli space** of all quotients of X with fundamental group π_1 (along with some other points?).

[1]

[2]

---

1. Benson Farb, Dan Margalit - A Primer on Mapping Class Groups, p.269 ↩
2. Benson Farb, Dan Margalit - A Primer on Mapping Class Groups, p.345 ↩