

Locally compact Hausdorff spaces

Definition. A Hausdorff space is **locally compact** if every point has a open neighborhood whose closure is compact.

2.4 Theorem Suppose K is compact and F is closed, in a topological space X . If $F \subset K$, then F is compact.

PROOF If $\{V_\alpha\}$ is an open cover of F and $W = F^c$, then $W \cup \bigcup_\alpha V_\alpha$ covers X ; hence there is a finite collection $\{V_{\alpha_i}\}$ such that

$$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. ////

Corollary If $A \subset B$ and if B has compact closure, so does A .

2.5 Theorem Suppose X is a Hausdorff space, $K \subset X$, K is compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \emptyset$.

PROOF If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q , such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \dots, q_n \in K$ such that

$$K \subset V_{q_1} \cup \cdots \cup V_{q_n}.$$

Our requirements are then satisfied by the sets

$$U = U_{q_1} \cap \cdots \cap U_{q_n} \quad \text{and} \quad W = V_{q_1} \cup \cdots \cup V_{q_n}. \quad \text{////}$$

Corollaries

- (a) Compact subsets of Hausdorff spaces are closed.
- (b) If F is closed and K is compact in a Hausdorff space, then $F \cap K$ is compact.

Corollary (b) follows from (a) and Theorem 2.4.

$[0, \infty]$ -measures and positive functions on \mathcal{C}_c

A locally finite positive Borel measure μ on X gives a functional

$$\Lambda : \mathcal{C}_c(X) \rightarrow k$$

$$\Lambda(f) := \int_{(X, \mu)} f$$

Thus functional is *positive* in the sense that if $f(X) \subset (0, 1)$ then $(\Lambda f)(X) \subset (0, 1)$

$$\text{locFinMes}(\mathfrak{B}_X, [0, \infty]) \rightarrow \mathcal{C}_c(X)^{*>0}$$

[1]

The converse of this statement is true for a much *regular* class of measures!

(Riesz representation of positive functionals on C_c) Let X be locally compact Hausdorff space and Λ be a positive linear functional on $C_c(X)$. Then there exists a σ -algebra \mathfrak{M}_Λ in X which contains all Borel sets \mathfrak{B}_X in X and there exists a **unique positive measure**

$$\mu : \mathfrak{M}_\Lambda \rightarrow [0, \infty]$$

such that $\Lambda(f) = \int_X f d\mu$ for every $f \in C_c(X)$, we have

$$\begin{aligned} \text{compact } K \subseteq X &\implies \mu(K) < \infty \\ E \in \mathfrak{M}_\Lambda &\implies \mu(E) = \inf \{ \mu(V) \mid E \subseteq V, V \text{ open } \subseteq X \} \\ \left\{ \begin{array}{l} E \text{ open} \\ \text{or} \\ E \in \mathfrak{M}_\Lambda, \mu(E) < \infty \end{array} \right. &\implies \mu(E) = \sup \{ \mu(K) \mid K \subseteq E, K \text{ cpt } \subseteq X \} \end{aligned}$$

and if $E \in \mathfrak{M}_\Lambda, A \subseteq E, \mu(E) = 0$ then $A \in \mathfrak{M}_\Lambda$.

Walter Rudin - Real and complex analysis-McGraw-Hill (1987) p 40

Thus

$$C_c(X, \mathbb{R})^{*>0} \cong_{\text{Set}} \text{RieszMes}(\mathfrak{B}_X, [0, \infty])$$

Borel \mathbb{C} -measures

(Riesz representation for Borel \mathbb{C} -measures) Let X be a locally compact Hausdorff space. Then every bounded linear functional

$$\varphi : C_0(X) \rightarrow \mathbb{C}$$

is represented by a unique regular complex Borel measure μ , that is,

$$\forall f \in C_0(X), \varphi(f) = \int_X f d\mu$$

Moreover, the norm of φ is the total variation of μ

$$\|\varphi\| = |\mu|(X)$$

$$C_0(X, \mathbb{C})^\star \cong_{\text{Ban}_\mathbb{C}} (\text{regMes}(\mathfrak{B}_X, \mathbb{C}), |\cdot|(X))$$

Borel $[0, 1]$ -measures

We have continuous maps

$$\text{ev}_f : C_0(X, \mathbb{C})^\star \rightarrow \mathbb{C}$$

so the subset of positive functionals

$$\mathcal{C}_0(X, \mathbb{C})^{\star>0} := \bigcap_{f \geq 0} \text{ev}_f^{-1}[0, \infty]$$

is an intersection of closed sets, so it is closed.

$$\mathcal{C}_0(X, \mathbb{C})^{\star>0} \cong \text{regMeas}(\mathfrak{B}_X, [0, \infty))$$

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1. math.uchicago.edu/~may/REU2023/REUPapers/Espejo.pdf#page=9.37 ↩