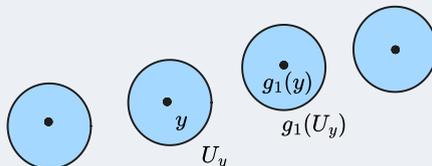# Covering space action

> **✈ Definition. Covering space action**
>
> Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space Y is a **covering space action** if **(orbits are evenly covered)** each $y \in Y$ has a neighborhood U_y such that
>
> $$\{g(U_y) | g \in \mathrm{im}\, G\} \text{ is a pairwise disjoint collection}$$
> $$\Longleftrightarrow g_1, g_2 \in \mathrm{im}\, G, g_1(U_y) \cap g_2(U_y) \neq \emptyset \implies g_1 = g_2$$
> $$\Longleftrightarrow g \in \mathrm{im}\, G, U_y \cap g(U_y) \neq \emptyset \implies g = \mathsf{Id}_Y$$
> $$\Longleftrightarrow |\{g \in \mathrm{im}\, G | U_y \cap g(U_y) \neq \emptyset\}| = 1$$

- covering action $\iff$ the action by $\mathrm{im}\, G$ is **free** and the action is **stably discontinuous** [1]
- On a Hausdorff space, by

> **☰ Let G be a group. An** *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ **on a Hausdorff topological space Y is finitely discontinuous**
>
> $$\Longleftrightarrow$$
>
> $\mathrm{im}\, G$ **has finite stabilizers and the action is stably discontinuous. Moreover, we can take the evenly covered set U_y for any $y \in Y$ to be $(\mathrm{im}\, G)_y$-stable.**

- - stably discontinuous and $\mathrm{im}\, G$ finite stabilizers $\implies$ finitely discontinuous.
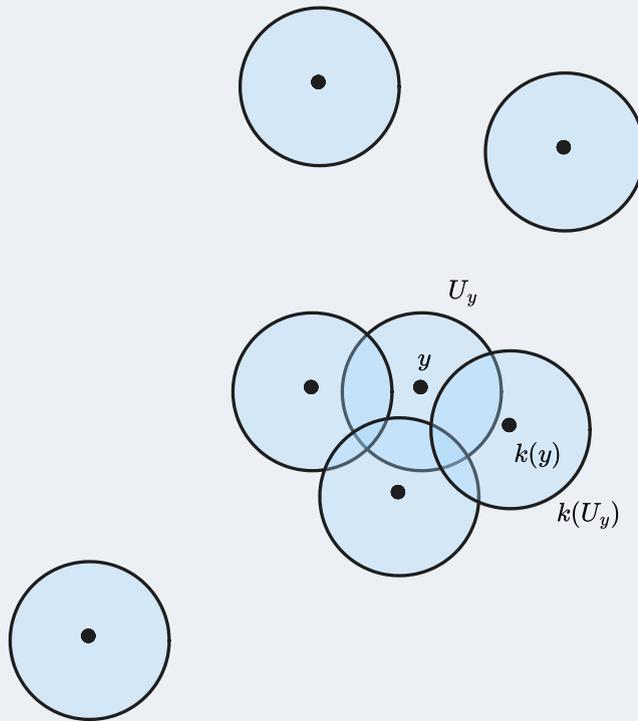
- - ☀ Let Y be Hausdorff and

$$G \leq \mathrm{Homeo}(Y)$$

be a faithful, finitely discontinuous action.
  - This means for all $y \in Y$ there is a nb U_y such that

$$\left| \underbrace{\{g \in G | U_y \cap g(U_y) \neq \emptyset\}}_{=:K_y} \right| < \infty$$
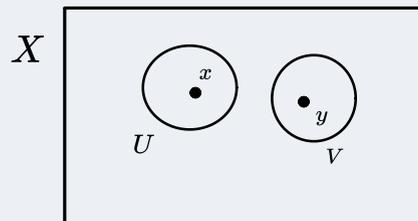
where $G_y \subseteq K_y$

U_y

y

$k(y)$

$k(U_y)$

- By

> ➤ Definition. T_2 **topological space or Hausdorff space**
>
> A <u>topological space</u> $(X, \mathfrak{T})$ is said to be T_2 **or Hausdorff** if *we can "house off" every pair of points using disjoint open sets*
>
> $$\forall x, y \in X \exists U, V \in \mathfrak{T}$$
> $$\text{st } x \in U, y \in V$$
> $$\text{and } U \cap V = \emptyset$$
>
> X
>
> x
>
> U
>
> y
>
> V

for every $y, k \in K_y \setminus G_y$ we have $U_k, V_k \subseteq Y$ such that

$$y \in U_k, k(y) \in V_k$$
$$U_k \cap V_k = \emptyset$$

- Consider

$$W := U_y \cap \bigcap_{k \in K_y \setminus G_y} U_k \cap k^{-1}(V_k)$$

- Then for $k_1 \in K_y$

$$W \subseteq U_k$$
$$\text{and } W := U_y \cap \bigcap_{k \in K_y \setminus G_y} U_k \cap k^{-1}(V_k)$$
$$k_1(W) = k_1(U_y) \cap \bigcap_{k \in K_y \setminus G_y} k_1(U_k) \cap k_1 k^{-1}(V_k)$$
$$\implies k_1(W) \subseteq V_{k_1}$$

- So for $k \in K_y \setminus G_y$

$$k(W) \subseteq V_k, W \subseteq U_k, V_k \cap U_k = \emptyset \implies k(W) \cap W = \emptyset$$

- Now, even for $g \in G \setminus K_y$ we have

$$W \subseteq U, g(W) \subseteq gU \implies g(W) \cap W = \emptyset$$

- Thus

$$\{k \in G_y | k(W) \cap W \neq \emptyset\} = G_y$$

covering $\iff$ the action by $\mathrm{im}\, G$ is **free** and the action is **finitely discontinuous**

> 📍 Definition. **Finitely discontinuous actions on topological spaces**
>
> Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space Y is **finitely discontinuous** if every point $y \in Y$ has a neighborhood U_y such that $U \cap g(U)$ is nonempty for only finitely many $g \in \mathrm{im}\, G$, that is,
>
> $$|\{g \in \mathrm{im}\, G | U_y \cap g(U_y) \neq \emptyset\}| < \infty$$

So a faithful *covering space action*

$$G \hookrightarrow \mathrm{Homeo}(Y)$$

in particular implies

- the orbits

$$G\{y\}$$

  are discrete
- stabilizers of a point fix everything

$$G_x = \{i_G\}$$

the quotient map is a normal covering

> **Proposition:** Let $G \to \mathrm{Homeo}(Y)$ be a covering space action. Then
>
> $$p : Y \to {}_G\backslash^Y$$
>
> is a normal covering space.

💡 For any $y \in Y$ let U_y be the onb of y such that
$$\{g(U_y)|g \in \text{im } G\}$$
is a pairwise disjoint collection.

- Thus
$$p^{-1}(p(U_y)) = \bigsqcup_{g \in \text{im } G} g(U_y)$$

- Also because p is a open continuous map
$$p : g(U_y) \to p(U_y)$$
is a homeomorphism for each $g \in G$.

- Thus p is a normal covering space.

📖 **Group action which is not a _covering action_ but the quotient map is a normal covering space.**

Consider
$$\mathbb{Z} \curvearrowright \mathbb{R} \coprod \mathbb{R}$$
$$\downarrow \text{Id}_{\mathbb{R}} \coprod \exp$$
$$\mathbb{R} \coprod S^1$$

where $\mathbb{Z}$ acts by Id by the first component and translations on the second component of $\mathbb{R} \coprod \mathbb{R}$
$$(x,1),(y,2) \overset{n}{\mapsto} (x,1),(y+n,2)$$

This action is not _free_ as stabilizer of $(x,1)$ is the whole group $\mathbb{Z}$.

Thus this action is **not** a covering action.

However, the map p is a normal covering map with $\mathbb{Z}$ being the whole covering automorphism group.

📙 **Theorem 12.14 (Covering Space Quotient Theorem).** _Let E be a connected, locally path-connected space, and suppose we are given an effective action of a group Γ on E by homeomorphisms. Then the quotient map $q : E \to E/\Gamma$ is a covering map if and only if the action is a covering space action. In this case, q is a normal covering map, and $\text{Aut}_q(E) = \Gamma$, considered as a group of homeomorphisms of E._

Proposition: Let
$$G \curvearrowright Y$$
is a faithful? **covering space action**, then

- if Y is path connected
$$G = \text{Aut}\,\text{Top}\left(Y \to {}^Y/_G\right)$$

- if Y is path connected and locally path connected

$$\operatorname{im} G \cong \frac{\pi_1\left(Y/G\right)}{\pi_*\pi_1(Y)}$$

on a topological manifold

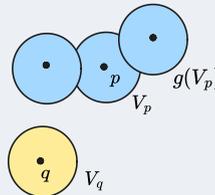### 🢒 Definition. coHausdorff actions

Let G be a group and

$$G \to \operatorname{Homeo}(Y)$$

be an *action by homeomorphisms* on a topological space Y. The action is called **coHausdorff** if it satisfies *either* of

- $\Gamma\backslash^X$ is Hausdorff
- if $p, q \in X$ are not in same Γ-orbit then there exists neighborhoods V_p of p and V_q of q such that

$$\forall g \in \Gamma, g(V_p) \cap V_q = \emptyset$$

Proposition: Let a **discrete** Lie group Γ acts **continuously and freely** on a topological manifold X. Then the action is **proper** $\iff$ it is a <u>covering space action</u> and coHausdorff $\iff$ it satisfies

🢒 Definition. discontinuosly coHausdorff actions on topological spaces

Let G be a group. An *action by homeomorphisms* $G \to \operatorname{Homeo}(Y)$ on a topological space Y is **discontinuosly coHausdorff** if it satisfies *any* of

- for each $x_1, x_2 \in X$ there exists respective nb $U_1, U_2 \subseteq X$ such that for every $\gamma \in \Gamma$

$$\gamma(U_1) \cap U_2 \neq \emptyset \iff \gamma(x_1) = x_2$$

- stably discontinuous and
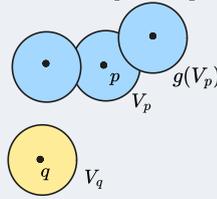
🢒 Definition. coHausdorff actions

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be an *action by homeomorphisms* on a topological space Y. The action is called **coHausdorff** if it satisfies *either* of

- $\Gamma\backslash^X$ is Hausdorff
- if $p, q \in X$ are not in same Γ-orbit then there exists neighborhoods V_p of p and V_q of q such that

$$\forall g \in \Gamma, g(V_p) \cap V_q = \emptyset$$

- 💡 free and proper $\implies$
  - quotient is Hausdorff
  - ...
- 💡 Let the action be coHausdorff and covering.
  - covering $\implies$ free
  - Let $\{g_i\} \subseteq \Gamma, \{p_i\} \subseteq X$ such that

  $$p_i \xrightarrow{i \to \infty} p, g_i(p_i) \xrightarrow{i \to \infty} p'$$

  - If p, p' are in different orbits, it contradicts coHausdorff property.
  - So there exists $g \in \Gamma$ such that $g(p) = p'$. This implies

  $$g^{-1} g_i(p) \xrightarrow{i \to \infty} p$$

  - By covering action property,

  $$g^{-1} g_i = \mathsf{Id}$$

  for large enough i.
  - Thus

  $$g = g_i$$

  which certainly converges. So the action is proper.

1.

> 🧭 Definition. **Stably discontinuous actions on topological spaces**
>
> Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space Y is **stably discontinuous** if every point $y \in Y$ has a neighborhood U_y such that
>
> $$\{g \in \mathrm{im}\, G | U_y \cap g(U_y) \neq \emptyset\} = (\mathrm{im}\, G)_y$$

↩