Discontinuous group actions on topological spaces

Definition. Stably discontinuous actions on topological spaces

Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space Y is **stably discontinuous** if every point $y \in Y$ has a neighborhood U_y such that

$$\{g \in \mathrm{im}\, G | U_y \cap g(U_y) \neq \emptyset\} = (\mathrm{im}\, G)_y$$

Definition. Finitely discontinuous actions on topological spaces

Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space Y is **finitely discontinuous** if every point $y \in Y$ has a neighborhood U_y such that $U \cap g(U)$ is nonempty for only finitely many $g \in \mathrm{im}\, G$, that is,

$$|\{g \in \mathrm{im}\, G | U_y \cap g(U_y) \neq \emptyset\}| < \infty$$

- finitely discontinuous $\implies$
  - action of $\mathrm{im}\, G$ has finite stabilizers
  - orbits are disjoint
- stably discontinuous and $\mathrm{im}\, G$ finite stabilizers $\implies$ finitely discontinuous.

Warning

Stably, or finitely discontinuous actions might not be **free**!

coHausdorff actions

Definition. discontinuosly coHausdorff actions on topological spaces

Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space Y is **discontinuosly coHausdorff** if it satisfies *any* of

- for each $x_1, x_2 \in X$ there exists respective nb $U_1, U_2 \subseteq X$ such that for every $\gamma \in \Gamma$

$$\gamma(U_1) \cap U_2 \neq \emptyset \iff \gamma(x_1) = x_2$$

- stably discontinuous and

Definition. coHausdorff actions

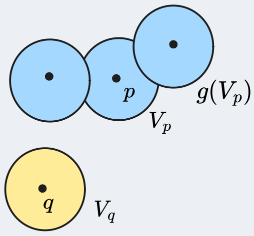
Let G be a group and

$$G \to \mathrm{Homeo}(Y)$$

be an *action by homeomorphisms* on a topological space Y. The action is called **coHausdorff** if it satisfies *either* of

- $\Gamma \backslash^{X}$ is Hausdorff
- if $p, q \in X$ are not in same Γ-orbit then there exists neighborhoods V_p of p and V_q of q such that

$$\forall g \in \Gamma, g(V_p) \cap V_q = \emptyset$$



for actions on Hausdorff spaces

Let Y be Hausdorff and

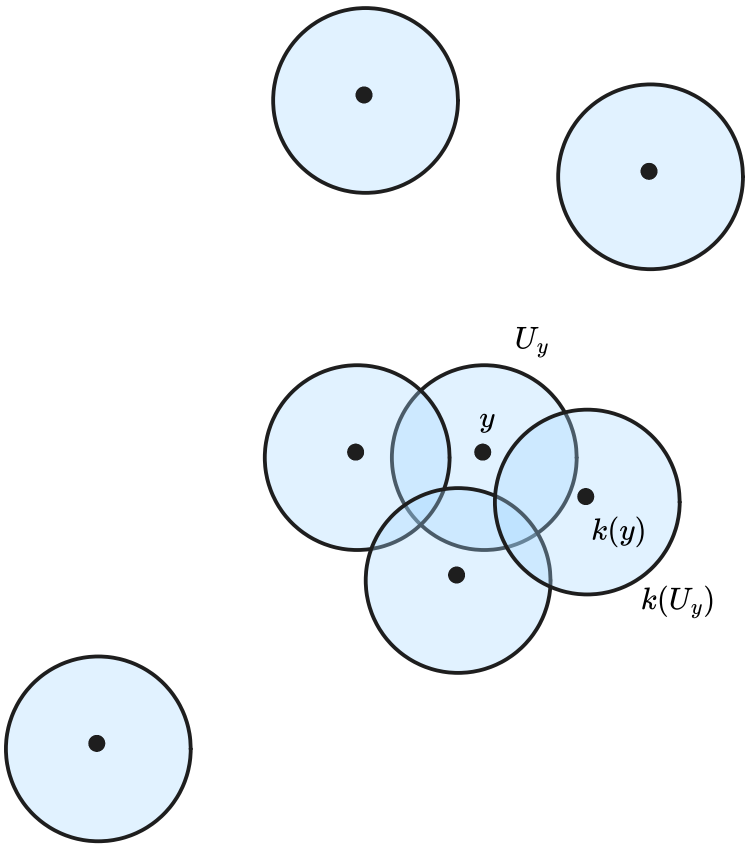
$$G \leq \mathrm{Homeo}(Y)$$

be a faithful, finitely discontinuous action.

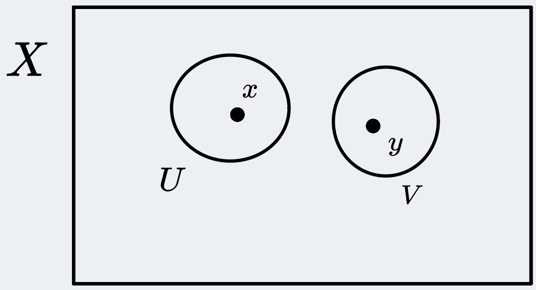
- This means for all $y \in Y$ there is a nb U_y such that

$$\left| \underbrace{\{g \in G | U_y \cap g(U_y) \neq \emptyset\}}_{=:K_y} \right| < \infty$$

where $G_y \subseteq K_y$



- By

> **Definition. T_2 topological space or Hausdorff space**
>
> A topological space $(X, \mathfrak{T})$ is said to be **T_2 or Hausdorff** if *we can "house off" every pair of points using disjoint open sets*
>
> $$\begin{aligned} \forall x, y \in X \exists U, V \in \mathfrak{T} \\ \text{st } x \in U, y \in V \\ \text{and } U \cap V = \emptyset \end{aligned}$$
>
> 
>

for every $y, k \in K_y \setminus G_y$ we have $U_k, V_k \subseteq Y$ such that

$$\begin{aligned} y \in U_k, k(y) \in V_k \\ U_k \cap V_k = \emptyset \end{aligned}$$

- Consider

$$W := U_y \cap \bigcap_{k \in K_y \setminus G_y} U_k \cap k^{-1}(V_k)$$

- Then for $k_1 \in K_y$

$$\begin{aligned} W &\subseteq U_k \\ \text{and } W &:= U_y \cap \bigcap_{k \in K_y \setminus G_y} U_k \cap k^{-1}(V_k) \\ k_1(W) &= k_1(U_y) \cap \bigcap_{k \in K_y \setminus G_y} k_1(U_k) \cap k_1 k^{-1}(V_k) \\ \implies k_1(W) &\subseteq V_{k_1} \end{aligned}$$

- So for $k \in K_y \setminus G_y$

$$k(W) \subseteq V_k, W \subseteq U_k, V_k \cap U_k = \emptyset \implies k(W) \cap W = \emptyset$$

- Now, even for $g \in G \setminus K_y$ we have

$$W \subseteq U, g(W) \subseteq gU \implies g(W) \cap W = \emptyset$$

- Thus

$$\{k \in G_y | k(W) \cap W \neq \emptyset\} = G_y$$

- To get a G_y-stable subset we can consider the set

$$\bigcap_{g \in G_y} gW$$

Let G be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a Hausdorff topological space Y is finitely discontinuous $\iff$ $\mathrm{im}\, G$ has finite stabilizers and the action is stably discontinuous. Moreover, we can take the evenly covered set U_y for any $y \in Y$ to be $(\mathrm{im}\, G)_y$-stable.

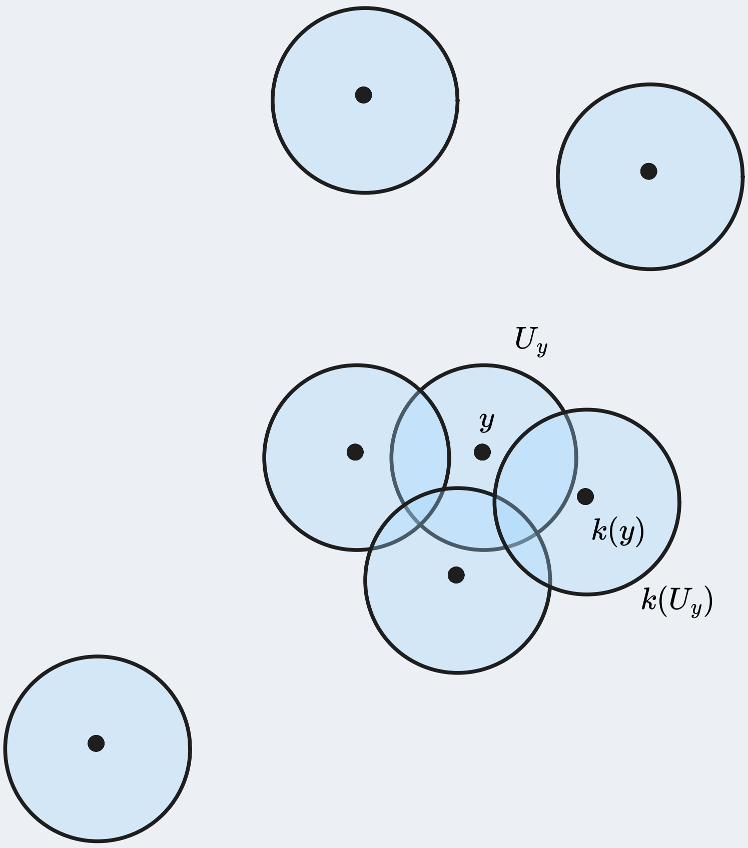
- - stably discontinuous and $\mathrm{im}\, G$ finite stabilizers $\implies$ finitely discontinuous.

- Let Y be Hausdorff and

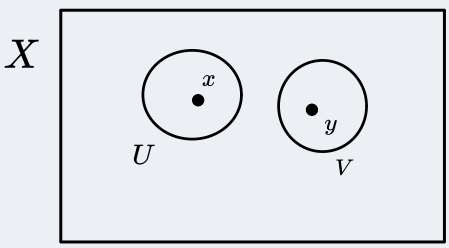
$$G \leq \mathrm{Homeo}(Y)$$

be a faithful, finitely discontinuous action.

> - This means for all $y \in Y$ there is a nb U_y such that
>
> $$\left| \underbrace{\{g \in G | U_y \cap g(U_y) \neq \emptyset\}}_{=:K_y} \right| < \infty$$
>
> where $G_y \subseteq K_y$



- By

> **Definition.** T_2 **topological space or Hausdorff space**
>
> A topological space $(X, \mathfrak{T})$ is said to be **T_2 or Hausdorff** if *we can "house off" every pair of points using disjoint open sets*
>
> $$\begin{aligned} \forall x, y \in X \exists U, V \in \mathfrak{T} \\ \text{st } x \in U, y \in V \\ \text{and } U \cap V = \emptyset \end{aligned}$$
>
> 
>

for every $y, k \in K_y \setminus G_y$ we have $U_k, V_k \subseteq Y$ such that

$$\begin{aligned} y \in U_k, k(y) \in V_k \\ U_k \cap V_k = \emptyset \end{aligned}$$

- Consider

$$W := U_y \cap \bigcap_{k \in K_y \setminus G_y} U_k \cap k^{-1}(V_k)$$

- Then for $k_1 \in K_y$

$$W \subseteq U_k$$
$$\text{and } W := U_y \cap \bigcap_{k \in K_y \setminus G_y} U_k \cap k^{-1}(V_k)$$
$$k_1(W) = k_1(U_y) \cap \bigcap_{k \in K_y \setminus G_y} k_1(U_k) \cap k_1 k^{-1}(V_k)$$
$$\implies k_1(W) \subseteq V_{k_1}$$

- So for $k \in K_y \setminus G_y$

$$k(W) \subseteq V_k, W \subseteq U_k, V_k \cap U_k = \emptyset \implies k(W) \cap W = \emptyset$$

- Now, even for $g \in G \setminus K_y$ we have

$$W \subseteq U, g(W) \subseteq gU \implies g(W) \cap W = \emptyset$$

- Thus

$$\{k \in G_y | k(W) \cap W \neq \emptyset\} = G_y$$

for continuous group actions, finitely discontinuous $\implies$ group is discrete

- Let

$$G \curvearrowright Y$$

be a continuous group action which is discontinuous, and let

$$g_i \xrightarrow{i \to \infty} i_G$$

be a sequence in G converging to identity.
  - Then

$$g_i(p) \xrightarrow{i \to \infty} p$$

by finitely discontinuous property, for $i \gg 1$

$$g_i(p) \in U_p \implies g_i \in \underbrace{K_p}_{\text{finite}}$$

  - Thus

$$g_i = i_G$$

for $i \gg 1$.

proper actions of a discrete group

Let Y be locally compact Hausdorff space and a topological group G acts on it continuously

$$\theta : G \curvearrowright Y$$

The action is finitely discontinuous and coHausdorff $\iff$ for each $p, q \in Y$ there exists nb U_p, U_q such that

$$|\{g \in G | \theta^g(U_q) \cap U_p\}| < \infty$$

$\iff$ G is discrete and the action is proper

[1]

Let

$$\pi_2 \times \theta : G \times Y \to Y \times Y$$

be proper and G be discrete.

- Then choose a cnb K_y of y. Then

$$(\pi_2 \times \theta)^{-1}(K_y \times K_y) = \{(g, x) | x \in K_y, gx \in K_y\}$$

$$\pi_1(\pi_2 \times \theta)^{-1}(K_y \times K_y) = \left\{ g \in G \middle| \underbrace{\exists x \in K_y : gx \in K_y}_{\substack{\iff g^{-1}K_y \cap K_y \neq \emptyset \\ \iff K_y \cap gK_y \neq \emptyset}} \right\}$$

is compact.

- Thus

$$|\{g \in G | K_y \cap gK_y \neq \emptyset\}| < \infty$$

- So an onb $U_y \subseteq K_y$ of y satisfies

$$|\{g \in G | U_y \cap gU_y \neq \emptyset\}| < |\{g \in G | K_y \cap gK_y \neq \emptyset\}| < \infty$$

- Thus the action θ is finitely discontinuous.

Proposition 12.24. *If a topological group G acts continuously and properly on a locally compact Hausdorff space E, then the orbit space E/G is Hausdorff.*

Proof. Let $\mathcal{O} \subseteq E \times E$ be the *orbit relation* defined in Problem 3-22. By the result of that problem, the orbit space is Hausdorff if and only if $\mathcal{O}$ is closed in $E \times E$. But $\mathcal{O}$ is just the image of the map $\Theta: G \times E \to E \times E$ defined by (12.5). Since E is a locally compact Hausdorff space, the same is true of $E \times E$, so it follows from Theorem 4.95 that Θ is a closed map. Thus the orbit relation is closed and E/G is Hausdorff. □

-

💡 Let $K \subseteq Y$ be compact and suppose

$$G_K$$

is infinite.

- For each $g \in G_K$ there is a $p \in K$ such that $g(p) \in K$, choosing one such point p define a map

$$\begin{aligned} F : G_K &\to K \times K \\ g &\mapsto (p, g(p)) \end{aligned}$$

- If $F(G_K)$ is finite, then there exists (p, x) such that

$$\{g \in G_K | x = g(p)\}$$

  is infinite. This is a contradiction to *finite stabilizers*, so $F(G_K)$ is infinite and has a limit point $(x_0, y_0) \in K \times K$.
- As $F(G_K) \subseteq \theta \times \pi_2(G \times X)$ which is closed, we have $(x_0, y_0) \in \theta \times \pi_2(G \times X)$, so there exists $g_0 \in G$ such that

$$x_0 = g_0(y_0)$$

- Let U be an evenly covered nb of y_0 and set $V = g_0 U$ which is a nb of x_0. This means

$$\{g \in G | g_0 U \cap gU \neq \emptyset\}$$

  is finite.
- As (x_0, y_0) is a limit point in $F(G_K)$, the nb $V \times U$ must contain infinitely many points of $F(G_K)$.
- But for each such $g \in G_K$ with

$$F(g) = (p, g(p)) \in V \times U$$

  we have

$$g(p) \in V \cap gU = g_0 U \cap gU$$

  as in

$$\underbrace{V \times U \cap F(G_K)}_{\text{infinite}} \subseteq \underbrace{\{g \in G | g_0 U \cap gU \neq \emptyset\}}_{\text{finite}}$$

  which is a contradiction.
- Thus G_K is finite for all compact K.

---

1. LeeTM 12-20 ↩