Closed map between topological spaces

Definition. Closed maps, maps with continuity of preimage

A map $f : X \to Y$ between topological spaces

- is **closed** if for every closed set $M \subseteq X$ the image $f(M) \subseteq Y$ is closed
- has **continuity of the preimage** if for all $y \in Y$ and open nb $U \subseteq X$ of $f^{-1}(y)$, there exists a open nb $V \subseteq Y$ of y such that

$$f^{-1}(V) \subseteq U$$

Inclusion maps $\iota : A \hookrightarrow X$ are closed $\iff$ for every closed subset $S \subseteq A$

$$\iota(S) = A \cap S$$

is closed $\iff$ $A \subseteq X$ is closed.

Projections

$$X \times Y \to X$$

are not closed in general.

Consider

$$\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Then

$$\underbrace{V(xy-1)}_{\text{closed}} \mapsto \underbrace{(-\infty, 0) \cup (0, \infty)}_{\text{not closed}}$$

Let Y be a topological space. Then Y is compact $\iff$ for every topological space X the projection

$$\pi_1 : X \times Y \to X$$

is closed.

Local homeomorphisms, even covering maps, are not closed in general.

Consider the covering

$$\begin{aligned} \mathbb{R} &\to S^1 \\ x &\mapsto \exp(2\pi i x) \end{aligned}$$

We know

$$f^{-1}(1) = \mathbb{Z}$$

and we have an evenly covered nb of $\mathbb{Z}$ mapping to 1

$$\bigsqcup_{m\in\mathbb{Z}} \left(\left(-\frac{1}{2}, \frac{1}{2}\right) + m\right) = f^{-1}(1)$$

Also note

$$\frac{1}{n+2} \xrightarrow{n\geq 1, n\to\infty} 0 \implies \exp\left(\frac{2\pi i}{n+2}\right) \xrightarrow{n\geq 1, n\to\infty} 1$$

Then, as this is an infinitely sheeted cover, we pick a preimage of each element of the converging sequence in *different* sheets

$$\frac{1}{|m|+2} + m \in \left(-\frac{1}{2}, \frac{1}{2}\right) + m$$

implying

$$\left\{\frac{1}{|m|+2} + m \middle| m \in \mathbb{Z}\right\}$$

is a discrete, closed set.

$$\underbrace{\left\{\frac{1}{|m|+2} + m \middle| m \in \mathbb{Z}\right\}}_{\text{closed}\subseteq\mathbb{R}} \mapsto \underbrace{\left\{\exp\left(\frac{2\pi i}{n+2}\right) \middle| n \in \mathbb{N}\right\}}_{\text{not closed}\subseteq S^1}$$

closed $\iff$ continuity of preimage

Let $f : X \to Y$ be map between topological spaces. Then f is closed $\iff$ f has continuity of the preimage.

Let f be closed.

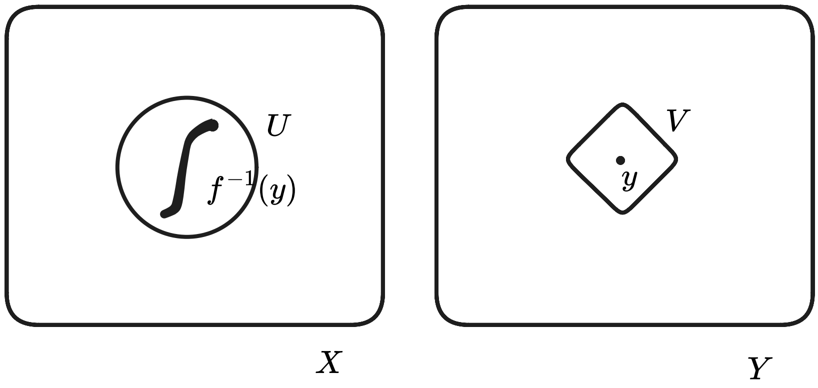
- For $y \in Y$ and U be a open nb of $f^{-1}(y)$. Then

$$f(X \setminus U)$$

is closed and does not contain y.

- Hence, there is a nb V of y such that

$$V \cap f(X \setminus U) = \emptyset \iff f^{-1}(V) \subseteq U$$

- 

Let $A \subseteq X$ be closed. Let $y \in Y \setminus f(A)$ then

$$f^{-1}(y) \subseteq X \setminus A$$

- As $X \setminus A$ is open nb of $f^{-1}(y)$, by *continuity of preimage*, we have a open nbd V of y such that

$$f^{-1}(V) \subseteq X \setminus A \iff V \cap f(A) = \emptyset$$

- Thus $f(A)$ is closed.

continuous map being closed is local in the codomain

Proposition: Let $f : X \to Y$ be continuous map between topological spaces.

- **(restriction)** Let $f : X \to Y$ is closed and let $V \subseteq Y$ is open, then

$$f|_{f^{-1}(V)} : f^{-1}(V) \to V$$

  is closed.
- **(gluing)** If for all $y \in Y$ there is an open nb V_y of y such that

$$f|_{f^{-1}(V_y)} : f^{-1}(V_y) \to V_y$$

  is closed, then $f : X \to Y$ is closed.
- **(being closed is local in the codomain)** The map $f : X \to Y$ is closed $\iff$ for every open cover $\{V_\alpha\}$ of Y, the restrictions

$$f|_{f^{-1}(V_\alpha)} : f^{-1}(V_\alpha) \to V_\alpha$$

  are closed.

[1]

⚠ **Warning**

The property of a continuous $f : X \to Y$ being closed is **not** local in the domain X.
Restrictions of Id_X to non-closed subsets $A \subset X$ is not closed.

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