Covering space of a topological space

Definition. Covering space of a topological space

A continuous map between topological spaces

$$\pi : Y \to X$$

is a **covering space map** if π is *surjective* and for any $x \in X$, $\exists U_x(\text{open}) \subseteq X$ of x such that

$$\pi^{-1}U_x = \bigsqcup_{a \in \pi^{-1}(x)} Y_a$$

where $\pi|_{V_a} : V_a \to U_x$ is a homeomorphism for each $a \in \pi^{-1}(x)$.

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and continuous map

$$h : Y \times [0,1] \to X$$

then for *any chosen lift $\tilde{f}$ of the initial* $h(-,0) := f_0 : Y \to X$

$$\tilde{f} : Y \to \tilde{X}, \pi \circ \tilde{f} = f$$

we have a unique lift of h

$$\exists!\ \tilde{h} : Y \times [0,1] \to \tilde{X},\ \pi \circ \tilde{h} = h$$

with the initial $\tilde{h}(-,0) = \tilde{f}$.

- Every path $h : \{\bullet\} \times [0,1] \to X$ can be uniquely lifted $\tilde{h} : \{\bullet\} \times [0,1] \to \tilde{X}$ once the initial point $\tilde{h}(\bullet, 0) \in \pi^{-1}(h(0)) \subseteq \tilde{X}$ has been chosen.
- Constant homotopies lift to constant homotopies.

- Let a homotopy $h : Y \times [0,1] \to X$ such that $\tilde{h}|_{A\times[0,1]}$ is a constant homotopy lifts to $\tilde{h} : Y \times [0,1] \to \tilde{X}$. Then $\tilde{h}$ restricts to the constant homotopy ?
  - Homotopy of paths rel to endpoints lifts to homotopy of paths rel to endpoints.

Definition. The universal cover of a topological space

Let (X, x_0) be a *globally and locally path-connected*, *semi-locally simply-connected* topological space with base point x_0. We define the **universal covering space of X** as the set of all homotopy classes (relative to endpoints) of paths in X starting at x_0

$$U(X, x_0) := \frac{\{\gamma : [0,1] \to X,\ \gamma(0) = x_0\}}{\text{homotopy rel to } \{0,1\}}$$

and give it the topology generated by

$$U_{[\gamma]} := \{[\gamma \cdot \eta] : \eta : [0,1] \to U, \eta(0) = \gamma(1)\}$$

for all path connected open sets $U \subseteq X$ for which $\pi_1(U) \to \pi_1(X)$ is trivial. With the map

$$\begin{aligned} p_{U(X,x_0)} : U(X) &\to X \\ [\gamma] &\mapsto \gamma(1) \end{aligned}$$

we define the **universal cover** as

$$p_{U(X)} : (U(X, x_0), [x_0]) \to (X, x_0)$$

and the universal **covering group action**

$$\begin{aligned} \pi_1(X, x_0) &\curvearrowright U(X, x_0) \\ [\gamma] &\mapsto \mathrm{Deck}(x_0, \tilde{\gamma}(1)) \end{aligned}$$

where $\tilde{\gamma}$ is the unique lift of the loop γ.

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is a simply-connected covering space.

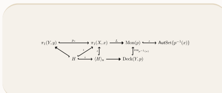
A simply-connected covering space of a topological space is therefore called a **universal cover**. It is unique up to isomorphism, so one is justified in calling it **the** universal cover. [1]

We have the *Galois correspondence* of covering spaces:

Correspondence between covering spaces and subgroups of the fundamental group

Let X just have a universal covering space? or path-connected, semi-locally simply connected required?

covering space (Y, y)	$p_*\pi_1(Y,y) =: H \leq$	surjective homomorphism to Deck group	lift homomorphism from fundamental group to Monodromy group	restriction homomorphism from Deck group to Monodromy group
any covering spaces (possibly not path-connected)			homomorphism $\pi_1(X,x) \to \mathrm{AutS}$ whose image is defined as $\mathrm{Mon}(p)$	

covering space (Y, y)	$p_*\pi_1(Y, y) =: H \leq$	**surjective homomorphism to Deck group**	**lift homomorphism from fundamental group to Monodromy group**	**restriction homomorphism from Deck group to Monodromy group**
n-sheeted covering spaces (possibly not path-connected)			conjugacy class of homomorphisms $\pi_1(X, x) \to S_n$ for n upto?	
if surjective then cover is path connected?			classes of **surjective** homomorphisms (cannot happen when $\pi_1(X, x)$ is Abelian for example)	
path-connected *based* covering spaces $p : (Y, y) \to (X, x$ that induces $p_* : \pi_1(Y, y) \hookrightarrow \pi$	subgroups $H := p_*\pi_1(Y, y)$ of $\pi_1(X, x_0)$	surjective homomorphism $\mathrm{N}(H) \to \mathrm{Deck}(Y,$ whose kernel is H that gives us $\frac{\mathrm{N}(H)}{H} \cong \mathrm{Deck}(Y$	$\pi_1(X, x) \to \mathrm{AutS}$	$\mathrm{res} : \mathrm{Deck}(Y, p) -$ is an *injective* group homomorphism. There *exists an unique* Deck transformation between y_1, y if $p_*\pi_1(Y, y_1) = p_*\pi$
	by *covering* group action through Deck transformations $\frac{U(X, x)}{H}$			
path-connected covering spaces (ignoring base points)	conjugacy classes of subgroups of $\pi_1(X, x)$			
$Y_1 \to Y_2 \to X$	$H_1 \leq H_2 \leq \pi_1(X$			

covering space (Y, y)	$p_*\pi_1(Y, y) =: H \leq$	surjective homomorphism to Deck group	lift homomorphism from fundamental group to Monodromy group	restriction homomorphism from Deck group to Monodromy group
normal covering spaces p of X that induces $p_* : \pi_1(Y, y) \hookrightarrow \pi$	*normal* subgroups $H = p_*\pi_1(Y, y)$ of $\pi(X, x_0)$	surjective homomorphism $\pi_1(X, x) \to \mathrm{Deck}($ whose kernel is $p_*\pi_1(Y, y)$ that gives us $\frac{\pi_1(X, x)}{H} \cong \mathrm{Deck}$	$\pi_1(X, x) \to \mathrm{AutS}$ is a group homomorphism whose kernel is H, hence $\frac{\pi_1(X, x)}{H} \cong \mathrm{Mon}$	$\mathrm{res} : \mathrm{Deck}(Y, p) \cong$ There exists a Deck transformation between any pair of elements in $p^{-1}(x)$.
	quotients by properly discontinuous *transitive* group action through Deck transformations			
n-cyclic cover	normal subgroup H such that $\frac{\pi_1(X, x)}{H} \cong \mathbb{Z}_n$	$\mathrm{Dec}(Y, p) \cong \mathbb{Z}_n$		
infinite cyclic cover	normal subgroup H such that $\frac{\pi_1(X, x)}{H} \cong \mathbb{Z}$	$\mathrm{Dec}(Y, p) \cong \mathbb{Z}$		
Abelian covering spaces	*normal* subgroups that *contain* the commutator subgroup	$\frac{\pi_1(X, x)}{H} \cong \mathrm{Deck}$ becomes an Abilean group!		$\mathrm{res} : \mathrm{Deck}(Y, p) \cong$ is an isomorpshism of Abelian groups

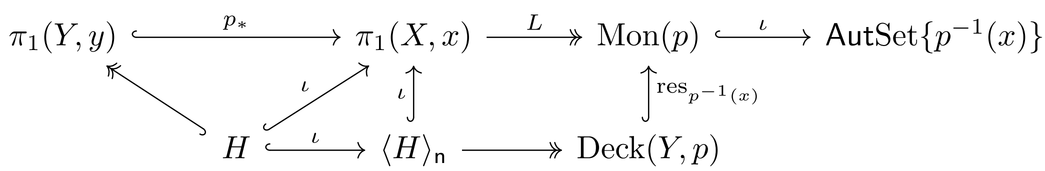
covering space (Y, y)	$p_*\pi_1(Y, y) =: H \leq$	surjective homomorphism to Deck group	lift homomorphism from fundamental group to Monodromy group	restriction homomorphism from Deck group to Monodromy group
Universal Abelian covering space	$H = [\pi_1(X, x), \pi_1$ the commutator subgroup	$\frac{\pi_1(X,x)}{H} \cong \text{Deck}$ becomes the largest Abilean (as a quotient) group!		$\text{res} : \text{Deck}(Y, p) \cong$ is an isomorpshism of Abelian groups
the *universal cover* whose fundamental group is 0	the subgroup is 0			
	$\frac{U(X,x)}{0}$			

Let $p : (Y, y) \to (X, x)$ be a path-connected covering space inducing $p_* : \pi_1(Y, y) \to \pi_1(X, x)$ then

$$\text{Deck}(Y, p) \cong \frac{\langle p_*\pi_1(Y, y)\rangle_{\mathsf{n}}}{p_*\pi_1(Y, y)}$$

and in particular if Y is a normal cover then H is a normal subgroup so

$$\text{Deck}(Y, p) \cong \frac{\pi_1(X, x)}{p_*\pi_1(Y, y)}$$



Definition. **The covering space functor**

$$\text{Covering}_X : \text{basedCovering}(X) \to \text{subGrp}\pi_1(X, x_0)$$

is a functor that maps

- objects:

$$p : (Y, y) \to (X, x_0) \mapsto p_*\pi_1(Y, y)$$

- morphisms:

$$\begin{array}{c} Y_1 \\ \Big\downarrow F \\ Y_2 \end{array} \quad \mapsto \quad p_*\pi_1(Y_1, y) \leq p_*\pi_1(Y_2, y)$$

Maybe

$$p \quad \begin{array}{c} (Y, y) \\ \downarrow \\ (X, x_0) \end{array} \mapsto p_* \quad \begin{array}{c} \pi_1(Y, y) \\ \downarrow \\ \pi_1(X, x_0) \end{array}$$

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1. hatcher-AT, page 77 ↩