Local homeomorphisms between topological spaces

Definition. Local homeomorphisms between topological spaces

Let X, Y be topological spaces. A map

$$f : X \to Y$$

is a **local homeomorphism** if for all $x \in X$ there is a onb U_x of x such that $f(U_x)$ is open in Y and

$$f|_{U_x} : U_x \to f(U_x)$$

is a homeomorphism.

- Let $f : X \to Y$ be a local homeomorphism.
  - Let $x \in X$ and V_x be an onb of $f(x)$. Let U_x be an onb of x such that $f(U_x)$ is open in Y

    $$f|_{U_x} : U_x \cong_{\mathsf{Top}} f(U_x)$$

    - Then $f(U_x) \cap V_x$ contains $f(x)$ is open in $f(U_x)$ and in Y.
    - Now, as $f|_{U_x}$ is continuous, the preimage $f^{-1}(f(U_x) \cap V_x)$ contains x and is open in U_x, so it is open in X.
    - Thus for every $x \in X$ and onb V_x of x, there is a onb $f^{-1}(f(U_x) \cap V_x)$ of x whose image is inside $f(V_x)$. Thus f is continuous at every $x \in X \implies f$ is continuous on X.
  - Let $U \subseteq X$ be open and let $y \in f(U)$.
    - For every $x \in f^{-1}(y) \cap U$ we have an onb $U_x \subseteq X$ such that $f(U_x)$is open in Y and

      $$f|_{U_x} : U_x \cong_{\mathsf{Top}} f(U_x)$$

    - Thus $f|_{U_x}$ is open, so $f(U_x \cap U)$ is open in $f(U_x)$ and in Y.
    - Now

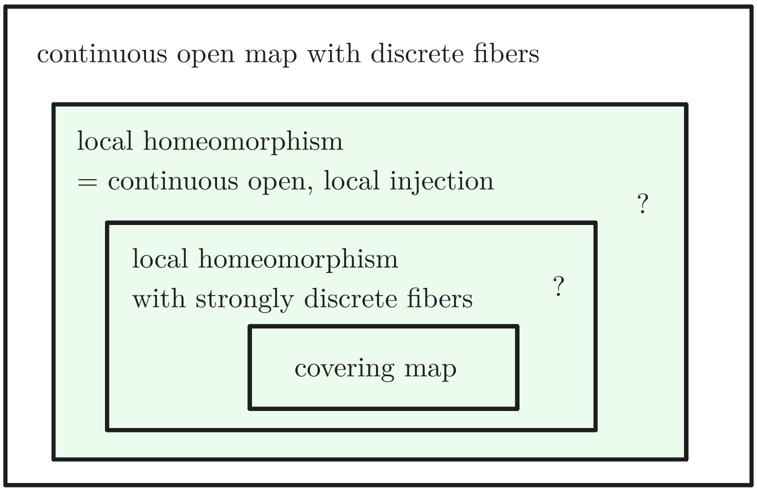
      $$f(U) = \bigcup_{x \in U} f(U_x \cap U)$$

      is open.
    - Thus, f is open.
  - Let $y \in Y$ and consider the fiber $f^{-1}(y)$.
    - For every $x \in f^{-1}(y)$ there is a onb U_x such that f is a bijection between U_x and $f(U_x)$.

- Thus U_x contains only one element of $f^{-1}(x)$

$$U_x \cap f^{-1}(y) = \{x\}$$

implying the fibers $f^{-1}(y)$ are a discrete subset of X.



local homeomorphisms between Hausdorff spaces

- Let X, Y be Hausdorff spaces and

$$f : X \to Y$$

be a local homeomorphism.