Proper maps between topological spaces

Definition. Proper, perfect, universally closed maps

A continuous map $f : X \to Y$ between topological spaces is

- **proper** if preimage of compact sets $K \subseteq Y$, $f^{-1}(K)$ are compact in X
- **Bourbaki-proper** if for any topological space Z the product map

  $$\mathrm{Id}_Z \times f : Z \times X \to Z \times Y$$

  is closed
- **perfect** if f is closed and has compact fibres ($f^{-1}(y)$ is compact for all $y \in Y$)
- **universally closed** if for every continuous map $g : Z \to Y$ the pullback

  $$g \times_Y f : Z \times_Y X \to Z$$

  is closed
- **sequential escaper** if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *escaping to infinity* then $\{f(x_n)\}_{n\in\mathbb{N}}$ *escapes to infinity*

[1]

[2]

[3]

Let $f : X \to Y$ be a continuous map between topological spaces. Then f is perfect $\implies$ f is proper.

Let $K \subseteq Y$ be compact and $\mathcal{O} = \{U_\alpha \subseteq X\}$ be an open cover of $f^{-1}(K)$.

- For each $p \in K$, $f^{-1}(p)$ is compact and $\mathcal{O}$ covers it, so it has a finite subcover

  $$\mathcal{O}_p \subseteq \mathcal{O}$$

- Let

  $$U'_p := \bigcup \mathcal{O}_p \supseteq f^{-1}(p)$$

  It is open in X. Thus $f(X \setminus U'_p)$ is closed in Y. So

  $$Y \setminus f(X \setminus U'_p)$$

  is open in Y.

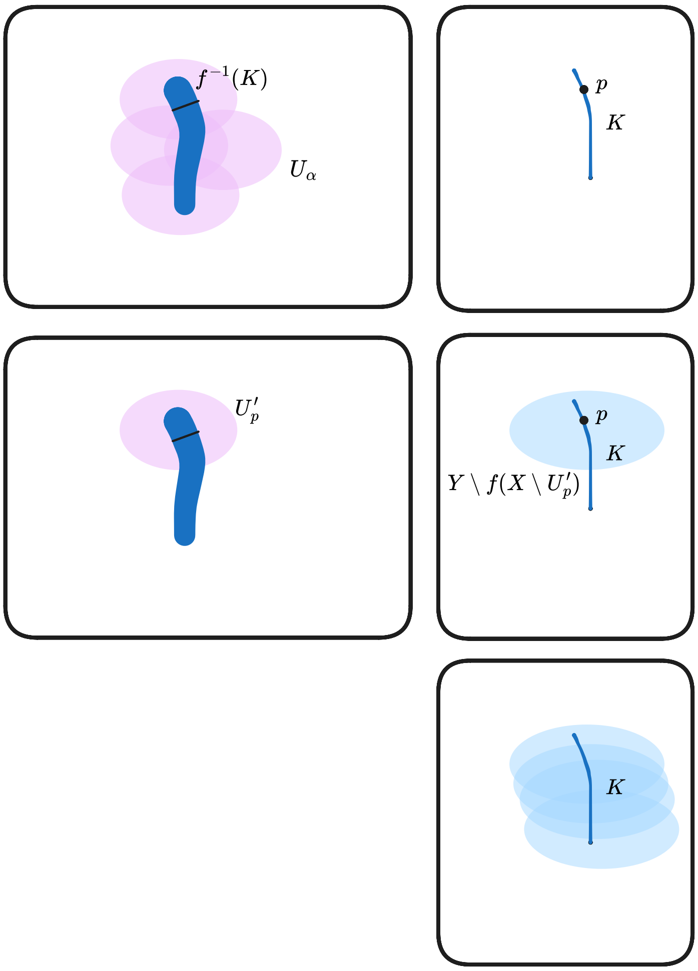
- $$\begin{aligned} q \in X \setminus U'_p \subseteq X \setminus f^{-1}(p) &\implies f(q) \neq p \\ &\implies p \notin f(X \setminus \ U'_p) \\ &\implies p \in Y \setminus f(X \setminus U'_p) \end{aligned}$$

- $$U'_{p_i} \subseteq Y \setminus f^{-1}(f(X \setminus U'_{p_i}))$$

- $$\{Y \setminus f(X \setminus U'_p) | p \in K\}$$

  is an open cover of K, thus has a finite subcover

  $$\{Y \setminus f(X \setminus U'_{p_i}) | 1 \leq i \leq N\}$$



- $$K \subseteq \bigcup_i Y \setminus f(X \setminus U'_{p_i})$$

- $$\begin{aligned} f^{-1}K &\subseteq \bigcup_i f^{-1}(Y \setminus f(X \setminus U'_{p_i})) \\ &= \bigcup_i \underbrace{Y \setminus \ f^{-1}(f(X \setminus U'_{p_i}))}_{\subseteq U'_{p_i}} \\ &= \bigcup_i U'_{p_i} \end{aligned}$$

which means

$$\bigcup_{i \leq N} \mathcal{O}_{p_i}$$

is a finite subcover of $\mathcal{O}$.

universally closed maps

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when are proper maps closed

Proposition: Let $f : X \to Y$ be a proper map which is either

- between two locally compact Hausdorff spaces
- between two metric spaces

Then f is closed

[5]

proper maps whose images are not closed

Definition. **A proper map $f : X \to Y$ is called bad-proper if the image $f(X) \subset Y$ is not closed.**

Proposition: Let $f : X \to Y$ be a bad-proper map. Then the inclusion

$$\iota : f(X) \hookrightarrow Y$$

is bad-proper.
Conversely, given a bad-proper inclusion $\iota : F \hookrightarrow Y$ and a continuous, proper surjection $X \twoheadrightarrow F$ gives a bad-proper

$$X \to Y$$

Definition. **A subset $F \subset X$ is bad-proper if the inclusion map $\iota : F \hookrightarrow X$ is bad-proper, that is, it is a proper map and $F \subset X$ is not closed.**

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1. Section 5.17 (005M): Characterizing proper maps—The Stacks project ↩
2. general topology - Definitions of proper maps - Mathematics Stack Exchange ↩
3. proper.pdf ↩
4. compactness-equivalence-and-application-to-proper-maps.pdf ↩

5. Proper_Maps.pdf ↩
6. proper_image_2.pdf ↩