When is a local homeomorphism between topological spaces a covering map

when domain is compact

Proposition: Let X, Y be Hausdorff spaces where X is compact and Y is connected. Then a local homeomorphism

$$f : X \to Y$$

is a covering map.

Let $f : X \to Y$ be a local homeomorphism.

- As local homeomorphism is open, we have $f(X)$ is both open and closed (because X is compact and Y is Hausdorff).
  - Thus by connectedness of Y we have $Y = f(X)$.
- As $y \in Y$ is closed, we have $f^{-1}(y)$ is a closed subset of compact $X \implies f^{-1}(y)$ is compact.
  - As f is a local homeomorphism, for every $x \in f^{-1}(y)$ there is a $U_x \subseteq Y$ such that

    $$f : U_x \to f(U_x)$$

    is a homeomorphism, in particular a bijection, so $f^{-1}(y) \cap U_x = \{x\}$.
  - Thus $f^{-1}(y)$ is compact and discrete so it is finite.
- Let

  $$f^{-1}(y) = \{x_1, \ldots, x_n\}$$

  where x_j are all distinct points of f.
  - We have onb U_{x_j} of each x_j such that

    $$f : U_{x_j} \to f(U_{x_j})$$

    is a homeomorphism.
  - Now consider

$$V_y := \left(\bigcap_{1 \le i \le n} \underbrace{f(U_n)}_{\text{open}} \right) \setminus f\left(X \setminus \bigcup_{1 \le i \le n} U_n \right)$$

surjective local homeomorphisms with constant finite fiber cardinality

Proposition: Let X, Y be Hausdorff spaces and

$$f : X \to Y$$

be a surjective local homeomorphism such that $|f^{-1}(y)| = n \in \mathbb{Z}_{>0}$ for all $y \in Y$ ($\iff$ fibers are compact and have constant cardinality). Then f is a finite covering map.

💡 Let

$$f^{-1}(y) = \{x_1, \ldots, x_n\}$$

- Let V_i be a onb of x_i such that $\{V_i\}$ is pairwise disjoint and $f|_{V_i}$ is a homeomorphism onto its image.
- Let

$$U_y := \bigcup_{1 \le i \le n} f(V_i)$$

- Then if $y_1 \in U_y := \bigcup_{1 \le i \le n} f(V_i)$ we have

$$f^{-1}(y_1) \cap \bigsqcup_{1 \le i \le n} V_i = \{v_1, \ldots, v_n\}$$

- But as $|f^{-1}(y_1)| = n$ then

$$f^{-1}(y_1) = \{v_1, \ldots, v_n\} \subseteq \bigsqcup_{1 \le i \le n} V_i$$

  for any $y_1 \in U_y$.
- Thus

$$f^{-1}(U_y) \subseteq \bigsqcup_{1 \le i \le n} V_i$$

[1]

when the map is proper

[2]

Proposition: Let X, Y be connected, locally path-connected and compactly generated Hausdorff spaces. Then a **proper** local homeomorphism

$$f : X \to Y$$

($\iff$ closed local homeomorphism with finite fibers) is a covering map.

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1. https://math.stackexchange.com/a/3967190 ↩
   11-9. Show that a proper local homeomorphism between connected, locally path-connected, and compactly generated Hausdorff spaces is a covering map.
2. 11-10. Show that a covering map is proper if and only if it is finite-sheeted.