When is a local homeomorphism between topological spaces a closed map

Local homeomorphisms, even covering maps, are not closed in general.

Consider the covering

$$\begin{aligned} \mathbb{R} &\to S^1 \\ x &\mapsto \exp(2\pi i x) \end{aligned}$$

We know

$$f^{-1}(1) = \mathbb{Z}$$

and we have an evenly covered nb of $\mathbb{Z}$ mapping to 1

$$\bigsqcup_{m\in\mathbb{Z}} \left(\left(-\frac{1}{2}, \frac{1}{2}\right) + m\right) = f^{-1}(1)$$

Also note

$$\frac{1}{n+2} \xrightarrow{n\geq 1, n\to\infty} 0 \implies \exp\left(\frac{2\pi i}{n+2}\right) \xrightarrow{n\geq 1, n\to\infty} 1$$

Then, as this is an infinitely sheeted cover, we pick a preimage of each element of the converging sequence in *different* sheets

$$\frac{1}{|m|+2} + m \in \left(-\frac{1}{2}, \frac{1}{2}\right) + m$$

implying

$$\left\{\frac{1}{|m|+2} + m \middle| m \in \mathbb{Z}\right\}$$

is a discrete, closed set.

$$\underbrace{\left\{\frac{1}{|m|+2} + m \middle| m \in \mathbb{Z}\right\}}_{\text{closed} \subseteq \mathbb{R}} \mapsto \underbrace{\left\{\exp\left(\frac{2\pi i}{n+2}\right) \middle| n \in \mathbb{N}\right\}}_{\text{not closed} \subseteq S^1}$$

A local homeomorphism with one infinite fibre *maybe* a closed map.

Consider the constant map

$$\mathbb{N} \to \{0\}$$

Then it is a local homeomorphism and a closed map.

We may extend the proof for exp to a general case by excluding existence of isolated points in the domain:

Let X be a separable metric space with no isolated points and Y be Hausdorff and first countable. Consider a local homeomorphism

$$f : X \to Y$$

such that there is a $y \in Y$ such that $f^{-1}(y)$ is infinite. Such a map f is not closed.

$f^{-1}(y)$ is a discrete infinite set.

As X is separable, any discrete infinite set is at most countable.

- So $f^{-1}(y) = \{x_n | n \in \mathbb{N}\}$ where x_n are all distinct.
- Pick $r_n > 0$ such that $B_{r_n}(x)$ are all pairwise disjoint and

$$f : B_{r_n}(x) \to f(B_{r_n}(x))$$

is a homeomorphism by

Let $S \subseteq X$ be discrete subset of a metric space X, that is, there is a collection $\{U_a\}_{a \in S}$ of open sets such that for all $a \in S$

$$U_a \cap S = \{a\}$$

Then there is a pairwise disjoint collection $\{V_a\}$ of subsets $V_a \subseteq U_a$ such that $V_a \cap S = \{a\}$, making S strongly discrete.

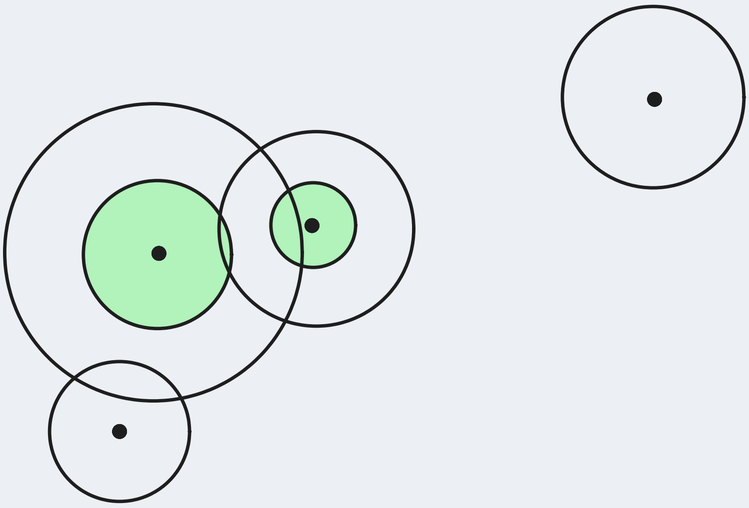
Choose $\epsilon(a) > 0$ such that $B_{\epsilon(a)}(a) \subseteq U_a$, which means

$$B_{\epsilon(a)} \cap S = \{a\}$$

- This means for $a, b \in S$

$$\begin{aligned} a \neq b \iff d(a,b) \geq \epsilon(a), \epsilon(b) \\ \iff d(a,b) \geq \max\{\epsilon(a), \epsilon(b)\} \\ \implies \\ d(a,b) < \max\{\epsilon(a), \epsilon(b)\} \implies a = b \end{aligned}$$

- Now consider $\{B_{\epsilon(a)/3}(a)\}_{a \in S}$



then

$$
\begin{aligned}
& y \in B_{\epsilon(a)/3}(a) \cap B_{\epsilon(b)/3}(b) \\
\implies & d(y,a) \le \frac{\epsilon(a)}{3}, d(y,b) \le \frac{\epsilon(b)}{3} \\
\implies & d(a,b) \le d(a,y) + d(y,b) \\
& \le \frac{1}{3}(\epsilon(a) + \epsilon(b)) \\
& \le \frac{2}{3} \max\{\epsilon(a), \epsilon(b)\} \\
& < \max\{\epsilon(a), \epsilon(b)\} \\
\implies & a = b
\end{aligned}
$$

Then pick some

$$z_n \in B_{r_n}(x_n) \cap f^{-1}(U_n)$$

where $\{U_n\}$ is a local basis of y so that $\{z_n\}$ is discrete closed set and

$$f(z_n) \xrightarrow{n\to\infty} y$$

- Thus

$$\underbrace{\{z_n\}}_{\text{closed}} \mapsto \underbrace{\{f(z_n)\}}_{\text{not closed}}$$

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1. https://math.stackexchange.com/a/4743954 ↩