Discrete subsets of topological spaces

Definition. Discrete subsets of topological spaces

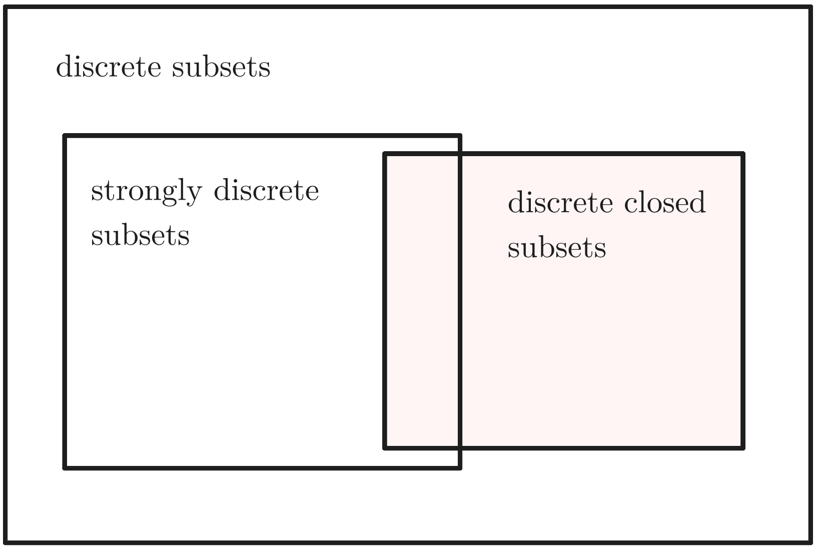
Let X be a topological space. Then a subset $S \subseteq X$ is **discrete** if it is a discrete space when equipped with the subspace topology $\iff$ for each $a \in S$ there is a onb $U_a \subseteq X$ of a such that $U_a \cap S = \{a\}$.

Definition. Strongly discrete subsets of topological spaces

Let X be a topological space. Then a subset $S \subseteq X$ is **strongly discrete** if there is a collection $\{U_a\}_{a \in S}$ of pairwise disjoint open sets such that

$$U_a \cap S = \{a\}$$

for each $a \in S$.



in metrizable spaces, discrete $\implies$ strongly discrete

Let $S \subseteq X$ be discrete subset of a metric space X, that is, there is a collection $\{U_a\}_{a \in S}$ of open sets such that for all $a \in S$

$$U_a \cap S = \{a\}$$

Then there is a pairwise disjoint collection $\{V_a\}$ of subsets $V_a \subseteq U_a$ such that $V_a \cap S = \{a\}$, making S strongly discrete.

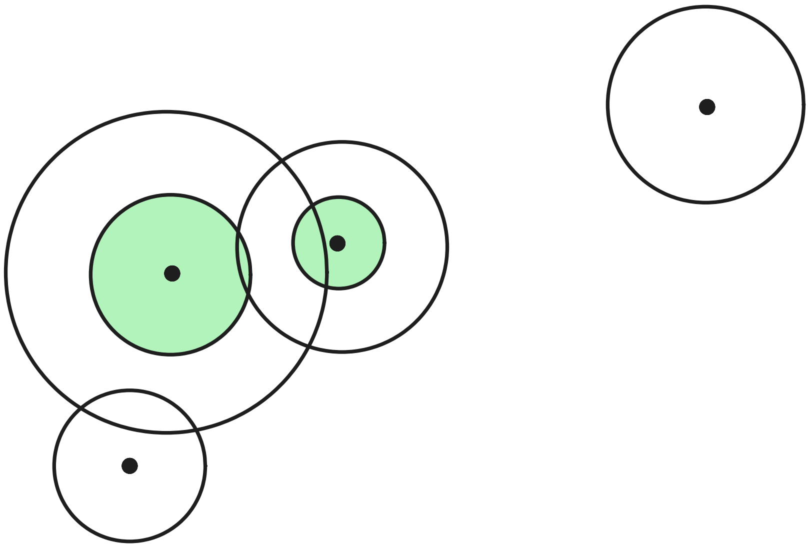
Choose $\epsilon(a) > 0$ such that $B_{\epsilon(a)}(a) \subseteq U_a$, which means

$$B_{\epsilon(a)} \cap S = \{a\}$$

- This means for $a, b \in S$

$$\begin{aligned} a \neq b &\iff d(a,b) \geq \epsilon(a), \epsilon(b) \\ &\iff d(a,b) \geq \max\{\epsilon(a), \epsilon(b)\} \\ &\implies \\ d(a,b) < \max\{\epsilon(a), \epsilon(b)\} &\implies a = b \end{aligned}$$

- Now consider $\left\{B_{\epsilon(a)/3}(a)\right\}_{a\in S}$



then

$$\begin{aligned} & y \in B_{\epsilon(a)/3}(a) \cap B_{\epsilon(b)/3}(b) \\ \implies & d(y,a) \leq \frac{\epsilon(a)}{3}, d(y,b) \leq \frac{\epsilon(b)}{3} \\ \implies & d(a,b) \leq d(a,y) + d(y,b) \\ & \leq \frac{1}{3}(\epsilon(a) + \epsilon(b)) \\ & \leq \frac{2}{3}\max\{\epsilon(a), \epsilon(b)\} \\ & < \max\{\epsilon(a), \epsilon(b)\} \\ \implies & a = b \end{aligned}$$

in Hausdorff spaces, finite subsets are strongly discrete

discrete closed subset S and locally finite collection of singletons $\{\{x\} | x \in S\}$