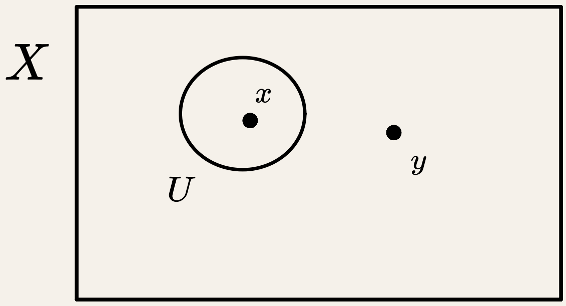
# Types of topological spaces

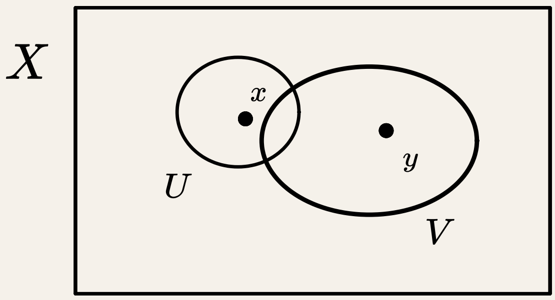
- **Definition.** $T_0$ **topological space**

  A topological space is said to be $T_0$ if for any pair $x, y \in X$, $\exists U \in \mathcal{T}$ such that $U$ contains one of them and not the other.


- **Definition.** $T_1$ **topological space**

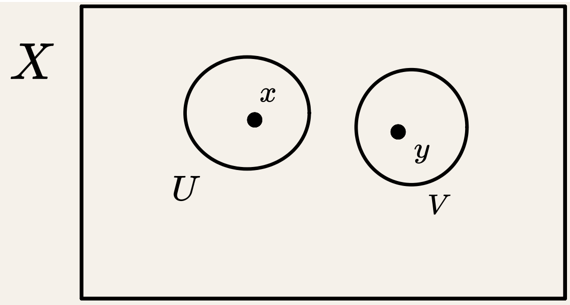
  A topological space is said to be $T_1$ is for any pair $x, y \in X$, $\exists U, V \in \mathcal{T}$ such that $U$ contains $x$ and not $y$ and $V$ contains $y$ but not $x$.


- **Definition.** $T_2$ **topological space or Hausdorff space**

  A topological space $(X, \mathfrak{T})$ is said to be **$T_2$ or Hausdorff** if *we can "house off" every pair of points using disjoint open sets*

$$\begin{aligned} &\forall x, y \in X \exists U, V \in \mathfrak{T} \\ &\text{st } x \in U, y \in V \\ &\text{and } U \cap V = \emptyset \end{aligned}$$

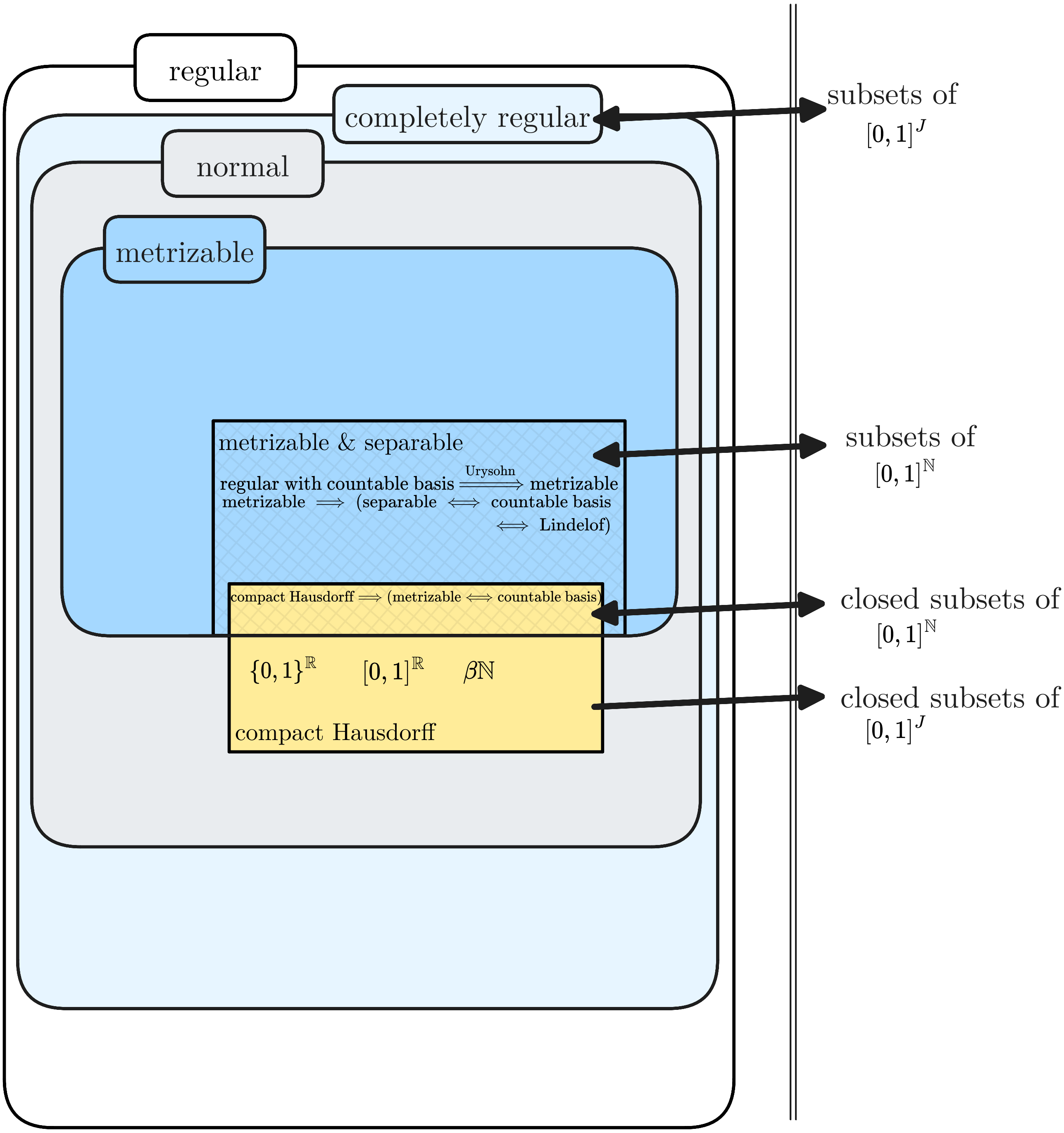



- T3 (regular)
- completely regular
- T4 (normal)
- T5
- ...
- **Definition.** A topological space $(X, \mathcal{T})$ is said to be first countable if every point in $X$ has a *countable local basis*.
- **Definition.** A topological space $(X, \mathcal{T})$ is called second countable or 2-countable if there is a countable basis on $X$ that generates $\mathcal{T}$.
- **Definition.** A topological space $(X, \mathcal{T})$ is called separable if there is a countable, dense subset of $X$.
- Compact
- Connected
- Path connected
  - Locally path connected
- Simply connected
  - Semi-locally simply connected
  - Locally simply connected
- **Definition.** A Hausdorff space is locally compact if every point has a open neighborhood whose closure is compact.

| | **1-countable** | **2-countable** | **countable dense subset (separable)** | **every open cover has a countable subcover (Lindelöf)** |
|---|---|---|---|---|
| 1-countable | | 2-countable $\implies$ 1-countable | | |
| 2-countable | | | | |
| countable basis | | | countable basis $\implies$ has a countable dense subset | countable basis $\implies$ Lindelöf |

**metrizable $\implies$ normal**
**normal $\implies$ regular**
**regular $\implies$ Hausdorff**

# Hausdorff topological spaces




**Proposition:** regular, countable basis $\Longrightarrow$ normal

compact Hausdorff $\Longrightarrow$ normal

well-order topology $\Longrightarrow$ normal

regular, Lindelöf $\Longrightarrow$ normal

**(Urysohn lemma)** Let $A, B \subseteq X$ be disjoint closed subsets of normal space $X$. Then there exists a continuous map

$$
\begin{aligned}
f : X &\to [a, b] \\
A &\mapsto a \\
B &\mapsto b
\end{aligned}
$$

- Let $X$ be a *normal space*.
  - *Choose $p, q \in X$*, because singletons closed subsets we have a *Urysohn function*

    $$f : X \to [0, 1], f(p) = 0, f(q) = 1$$

    that separates these points.
    - If $X$ is connected and not singleton then we have a surjective map from $X$ to $[0, 1]$, so $X$ is uncountable.

**(Urysohn metrization theorem) regular space with a countable basis $\implies$ homeomorphic to a subset of $[0, 1]^{\mathbb{N}}$ $\implies$ metrizable**

**metrizable $\implies$ (separable $\iff$ second countable)**

**compact metric space $\implies$ second countable, separable Hausdorff**

[1]

**Corollary:** Since

**compact metric space $\implies$ second countable, separable Hausdorff**

[1]

---

1. https://planetmath.org/acompactmetricspaceissecondcountable ↩

and

**(Urysohn metrization theorem) regular space with a countable basis $\implies$ homeomorphic to a subset of $[0,1]^{\mathbb{N}}$ $\implies$ metrizable**

This implies: compact Hausdorff space $\implies$ (metrizable $\iff$ it is second-countable).

---

Definition. **A topological space of $\sigma$-compact if its a countable union of compact subsets.**

**locally compact, second countable Hausdorff $\implies$ $\sigma$-compact**

[2]

**separable, locally compact, metrizable $\implies$ $\sigma$-compact**

**$\sigma$-compact metrizable $\implies$ separable**

**Corollary:**

**separable, locally compact, metrizable $\implies$ $\sigma$-compact**

and

**$\sigma$-compact metrizable $\implies$ separable**

implies compact metrizable $\implies$ (separable $\iff$ $\sigma$-compact)

**locally compact metric space $\implies$ (admits a proper/Heine-Borel metric**
**$\iff$ second countable**
**$\iff$ $\sigma$-compact**
**$\iff$ Lindelof)**

ProperMetricSpaces.pdf

## compactification

Definition. **Compactification of a space $X$**

A **compactification** of a space $X$ is a **compact Hausdorff** space $Y$ with

$$\iota : X \hookrightarrow Y, \overline{\iota(X)} = Y$$

If $X$ has a compactification, then $X$ must be completely regular.

---

1. https://planetmath.org/acompactmetricspaceissecondcountable ↩
2. https://math.stackexchange.com/questions/57348/hausdorff-locally-compact-and-second-countable-is-sigma-compact ↩