When a group plays ping pong

(Ping-pong lemma for F_2) Let $\Gamma = \langle a, b\rangle$ for $a, b \in \Gamma$ and there is a action $\Gamma \curvearrowright X$ such that there are non-empty $A, B \subseteq X$ with $B \nsubseteq A$ such that for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$a^n(B) \subseteq A, b^n(A) \subseteq B$$

Then $\Gamma \cong F_2$ and Γ is freely generated by $\langle a, b\rangle$.

Suppose the map

$$\langle x, y\rangle \twoheadrightarrow \Gamma$$

is not injective. So there is a word $w(x, y) \in \langle x, y\rangle \setminus \{\emptyset\}$ such that $w(a, b) = 1_\Gamma$. Depending on the first and last letter of $w(x, y)$ we have four cases:

- The word $w(x, y)$ starts and ends with x

  $$w(x, y) = x^{n_0}y^{m_1} \dots y^{m_k}x^{n_k}$$

  Then

  $$\begin{aligned} B = 1_\Gamma(B) &= w(x, y)(B) \\ &= a^{n_0}\underbrace{b^{m_1} \dots \underbrace{b^{m_k}\underbrace{a^{n_k}(B)}_{\subseteq A}}_{\subseteq B}}_{\subseteq B} \\ &\subseteq A \end{aligned}$$

  which contradicts $B \nsubseteq A$.
- ...
- Hence, the surjective map $\langle x, y\rangle \twoheadrightarrow \Gamma$ is also injective, making $\langle x, y\rangle \cong \Gamma$ such that $x, y \mapsto a, b$.