Subgroups of finite index of a finitely generated group

- Let Γ be a finitely generated group and $N \geq 1$. Let H be a subgroup of finite index

$$H \leq \Gamma, [\Gamma : H] = N$$

  - Thus Γ acts on the coset space of cardinality N

$$\Gamma \curvearrowright {}^{\Gamma}\!/_{H}$$

    where the stabilizer of H is precisely H.
  - That is, when the coset space is given an enumeration

$$\begin{aligned} {}^{\Gamma}\!/_{H} &\cong \{1, \ldots, N\} \\ H &\mapsto 1 \end{aligned}$$

    we have a homomorphism

$$\varphi : \Gamma \to S_N$$

    where

$$H = \{g \in \Gamma | \varphi^g(1) = 1\} = \mathrm{stab}_\varphi(1)$$

Hence,

- every subgroup of Γ of index $\leq N$ occurs as a stabilizer of 1 of some group action on $\{1, \ldots, N\}$

$$\{H \leq \Gamma | [\Gamma : H] \leq N\} \subseteq \{\mathrm{stab}_\varphi(1) | \varphi \in \mathrm{Hom}\mathsf{Grp}(\Gamma, S_N)\}$$

  - $$\begin{aligned} |\{H \leq \Gamma | [\Gamma : H] \leq N\}| &\leq |\mathrm{Hom}\mathsf{Grp}(\Gamma, S_N)| \\ &\leq (N!)^{r(\Gamma)} \end{aligned}$$

Definition.

$$N \mapsto |\{H \leq \Gamma | [\Gamma : H] \leq N\}|$$

- Let H be a subgroup of finite index

$$H \leq \Gamma, [\Gamma : H] = N$$

  - Then consider the action

$$\begin{aligned} \Gamma \curvearrowright {}^{\Gamma}\!/_{H} &\cong \{1, \ldots, N\} \\ H &\mapsto 1 \end{aligned}$$

- For any $g \in \Gamma$ we must have some repetition in

$$\underbrace{1, g(1), \ldots, g^{N-1}(1)}_{N \text{ elements}}, g^N(1)$$

thus

$$\begin{aligned}\exists k \in \{0, \ldots, N-1\} : g^N(1) = g^k(1) \\ \implies g^{N-k}(1) = 1 \\ \implies \exists a \in \{1, \ldots, N\} : g^a \in \mathrm{stab}(1) = H\end{aligned}$$

$$\sum_{n \in \mathbb{N}} \frac{|\mathrm{HomGrp}(\Gamma, S_n)|}{n!} z^n = \exp\left(\sum_{n \in \mathbb{N}} \frac{|\{H \leq \Gamma | [\Gamma : H] = n\}|}{n} z^n\right)$$

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1. The answer to the puzzle | Annoying Precision ↩