Interpretation of measure preserving endomorphisms

X	$x \in X$	
$B \subseteq X$	$x \in B$	
χ_B	$\chi_B(x)$	$\mathtt{check}(x \in B) \in \{0,1\}$
μ	$\mu(B)$	$\mu(B) = \mu\text{–}\mathtt{P}(x \in B)$ is the **size** of B with respect to μ is the **probability** of randomly picking a point x in B
$T : X \to X$ that is μ-invariant	Definition. **Measure preserving endomorp** Let $(X, \mathfrak{M}, \mu)$ be a $[0,\infty]$-measure space and $$T : X \to X$$ be a measurable map. Then T *preserves the measure μ* that is μ is a **T-invariant measure** or **T is μ-preserving transformation** if $$A \in \mathcal{A} \implies \mu(T^{-1}(A))$$	$\mu(T^{-1}(A)) = \mu(A)$ the **size** of *all* points going into A after one iteration is same as the **size** of A
		$\mu\text{–}\mathtt{P}(x \in B) = \mu\text{–}\mathtt{P}(Tx \in B)$ **probability** of picking x which is in B after an iteration $=$ **probability** of picked x in B
	$f : X \to \mathbb{R}$	*observable*

	$U_T(f) := f \circ T : X \to \mathbb{R}$	
	$\mathfrak{X}_T^{(n)} := f \circ T^n : X \to \mathbb{R}$ a stochastic process?	
	$\int_X f$	$\mu\text{–}\mathtt{E}(f)$ *expectation value*
	$\mu(B) = \int_X \chi_B \mathrm{d}\mu$	$\mu(B) = \mu\text{–}\mathtt{E}(\mathtt{check}(- \in B))$
$\chi_B \circ T^k$	$\chi_B \circ T^k(x)$	$\mathtt{check}(T^k(x) \in B) \in \{0, 1\}$
		$\mathtt{returns}(B) := \{x \in B \mid \exists n > 0$
	- Hence we conclude $\mu(A) > 0 \implies \{T^{-n}(A)$ which is $\iff$ there exists $n, m \in \mathbb{N}, n > m$ $x \in T^{-n}(A) \cap T^{-m}(A)$ that is there exists points in A return back to A.	$\mu(B) > 0 \implies \mathtt{returns}(B) \neq$ or $\mathtt{returns}(B) = \emptyset \implies \mu(B) =$

	- Consider the set $A \subset B$ of points that do not return to B $A := \{x \in B \mid \forall n > 0 : T$, so this means points of A do not return to A either. If $T^{-n}(A)$ are *not* pairwise disjoint for all $n \in \mathbb{N}$ then there exists $n, m \in \mathbb{N}, n > m$ $x \in T^{-n}(A) \cap T^{-m}(A)$ meaning the point $T^n(x)$ in A returns to A, which contradicts the definition of A. Thus $T^{-n}(A)$ are all disjoint.	$\texttt{returns}(B \setminus \texttt{returns}(B)) = \emptyset$
	(weak form of Poincare recurrence) **Let $(X, \mathcal{A}, \mu)$ be a probability space and $T : X \to X$ be a μ-preserving transformation and $B \subseteq X$ such that $\mu(B) > 0$. Then the set of points in B that never returns to B has measure zero.**	$\mu(B) > 0$ $\implies \mu(B \setminus \texttt{returns}(B)) = 0$

(infinite Poincare recurrence) **Let $(X, \mathcal{A}, \mu)$ be a probability space and $T : X \to X$ be a μ-preserving transformation and $B \subseteq X$ such that $\mu(B) > 0$. Then the set of points in B that returns to B *only finitely many times* has measure zero.**

- Now consider the set of points in B that do not return to B *infinitely many times* that is

  $$A_\infty := \{x \in B \mid$$

  > - If we consider the sets
  >
  > $$A := \{x \in$$
  > $$A_k := \{x \in$$
  >
  >   for $k \geq 1$,
  > - We see
  >
  > $$\iff T^k(x$$

so

$$A_k = T^{-k}($$

for all $k \geq 1$.

- If $x \in A_\infty$ there must be a largest integer k such that $T^k(x) \in B$ so $x \in A_k$. Conversely, if $x \in B \cap A_k$, we know

  $$T^n(x) \notin B$$

  for all $n > k$ meaning $x \in A_\infty$.
- Thus

  $$A_\infty = B \cap$$

  implying

  $$\implies \mu(A_\infty$$

- And we have already proven $\{A_k \mid k \in \mathbb{N}$

are all disjoint

- Consider the set A of points that do not return to

H A, so this means points of A do not return to A eithe

r. If T are *not* pairwise disjoint for all n then there e exists n x neaning the point T in A returns to A, whic h contradicts the definition of A. Thus T are all dis joint.

- Let $(X, \mathcal{A}, \mu)$ be a **probability space** and $T : X \to X$ be a μ-preserving transformation. For a measurable subset $A \subseteq X$, consider the subsets

  $$A, T^{-1}(A), T^{-2}($$

  which are all measurable subsets of X.
  - As

    $$T^{-n}(A) \subseteq$$

    we have

    $$\implies \mu\Big(\bigcup_{n\in}$$

  - But as $\mu(A) = \mu(T$, if $T^{-n}(A)$ are all disjoint sets we have

$$\mu(X) \geq \sum_{n\geq}$$

- Hence we conclude

  $$\mu(A) > 0$$

  which is $\iff$ there exists $n, m \in \mathbb{N}, n$

  $$x \in T^{-n}(A$$

  that is there exists points in A return back to A.
- The *contrapositive* of the above statement is

  $$\{T^{-n}(A) \mid$$

- Thus we can conclude by the previous argument

  $$\mu(A_k) = 0 \implies$$

ergodic	Definition. **Ergodic endomorp** A μ-preserving transformation $T : X \to X$ on a measure space $(X, \mathcal{A}, \mu)$ is called ***μ*-ergodic** if $A \in \mathcal{A}, T^{-1}(A) = A \implies$	
strongly mixing	Definition. **Strongly mixing endomorp** A μ-preserving transformation $T : X \to X$ on a **probability** space $(X, \mathcal{A}, \mu)$ is called **strongly mixing** if $A, B \in \mathcal{A} \implies \mu(A \cap T$	$\mu{-}\mathtt{P}(x \in A \& T^n x \in B) \to \mu(A)\mu$