Measure preserving endomorphisms

Definition. Measurable endomorphisms

non-singular	$$\mu(A) = 0 \iff \mu(T^{-1}A) = 0$$
conservative	if $\mu(A) > 0$ then there exists $n \in \mathbb{N}$ such that $$\mu(T^{-n}A \cap A) > 0$$ $\iff$? every wandering set has measure zero
dissipative	if it is not conservative
totally dissipative	for every $A \in \mathfrak{M}$ $$\mu\left\{x \in A \vert T^n(x) \in A \text{ infinitely often}\right\} = 0$$
incompressible	if $A \subseteq T^{-1}A$ then $$\mu(T^{-1}A) = \mu(A)$$

Definition. Measure preserving endomorphisms

Let $(X, \mathfrak{M}, \mu)$ be a $[0, \infty]$-measure space and

$$T : X \to X$$

be a measurable map. Then T *preserves the measure μ* that is μ is a **T-invariant measure** or **T is μ-preserving transformation** if

$$A \in \mathcal{A} \implies \mu(T^{-1}(A)) = \mu(A)$$

Definition. Ergodic endomorphisms

A μ-preserving transformation $T : X \to X$ on a measure space $(X, \mathcal{A}, \mu)$ is called **μ-ergodic** if

$$A \in \mathcal{A}, T^{-1}(A) = A \implies \mu(A) = 0 \text{ or } \mu(X \setminus A) = 0$$

Definition. Incompressible transformation on a measure space

A measurable function

$$T : X \to X$$

on a measure space $(X, \mathcal{A}, \mu)$ is called **incompressible** if

$$A \in \mathcal{A}, A \subseteq T^{-1}(A) \implies \mu(T^{-1}(A)) = \mu(A)$$

Definition. Strongly mixing endomorphisms

A μ-preserving transformation $T : X \to X$ on a **probability** space $(X, \mathcal{A}, \mu)$ is called **strongly mixing** if

$$A, B \in \mathcal{A} \implies \mu(A \cap T^{-n}(B)) \xrightarrow{n\to\infty} \mu(A)\mu(B)$$

ergodic	**for every $A \in \mathfrak{M}$ with $\mu(A) > 0$ there is a $n \geq 1$ such that** $$\mu(T^n(A) \cap A) > 0$$

	if for all $A, B \in \mathfrak{M}$ we have
strongly mixing	$$\mu(A \cap T^{-n}B) \xrightarrow{n\to\infty} \mu(A)\mu(B)$$
weakly mixing	$$\sum_{n=0}^{N-1} \frac{1}{N} \lvert \mu(A \cap T^{-n}B) - \mu(A)\mu(B) \rvert \xrightarrow{N\to 0} 0$$

Strong mixing on a probability space $\implies$ ergodicity.

Suppose $A \in \mathcal{A}$ such that $T^{-1}(A) = A$. Then

$$\mu(E) = \mu(E \cap T^{-n}(E)) \xrightarrow{n\to\infty} \mu(E)^2$$

which means

$$\mu(E)^2 = \mu(E) \implies \mu(E) = 1 \text{ or } 0$$