

Recurrence in iterations of a measure preserving endomorphism

[1]

- Let (X, \mathcal{A}, μ) be a **probability space** and $T : X \rightarrow X$ be a μ -preserving transformation. For a measurable subset $A \subseteq X$, consider the subsets

$$A, T^{-1}(A), T^{-2}(A), \dots$$

which are all measurable subsets of X .

- As

$$T^{-n}(A) \subseteq X$$

we have

$$\begin{aligned} & \bigcup_{n \in \mathbb{N}} T^{-n}(A) \subseteq X \\ \implies & \mu\left(\bigcup_{n \in \mathbb{N}} T^{-n}(A)\right) \leq \mu(X) < \infty \end{aligned}$$

- But as $\mu(A) = \mu(T^{-n}(A))$, if $T^{-n}(A)$ are all disjoint sets we have

$$\mu(X) \geq \sum_{n \geq 0} \mu(T^{-n}(A)) = \sum_{n \geq 0} \mu(A)$$

- Hence we conclude

$$\mu(A) > 0 \implies \{T^{-n}(A) \mid n \in \mathbb{N}\} \text{ are not disjoint}$$

which is \iff there exists $n, m \in \mathbb{N}, n > m$

$$x \in T^{-n}(A) \cap T^{-m}(A) \implies T^n(x) \in AT^{m-n}(T^n x) = T^m(x) \in A$$

that is there exists points in A return back to A .

- The *contrapositive* of the above statement is

$$\{T^{-n}(A) \mid n \in \mathbb{N}\} \text{ are disjoint} \implies \mu(A) = 0$$

- Thus for a subset $B \subseteq X$ with $\mu(B) > 0$, there are points in $B \cap T^{-n}(B)$ for some n , that means there are points in B that return back to B after some iterations.
- Consider the set $A \subset B$ of points that do not return to B


$$A := \{x \in B \mid \forall n > 0 : T^n(x) \notin B\}$$

, so this means points of A do not return to A either. If $T^{-n}(A)$ are *not* pairwise disjoint for all $n \in \mathbb{N}$ then there exists $n, m \in \mathbb{N}, n > m$

$$x \in T^{-n}(A) \cap T^{-m}(A) \implies T^n(x) \in A, T^{m-n}(T^n x) = T^m(x) \in A$$

meaning the point $T^n(x)$ in A returns to A , which contradicts the definition of A . Thus $T^{-n}(A)$ are all disjoint.

Thus applying the same conclusion for the set of points in B that do *not* return back to B we show that it is a set of measure zero:

 **(weak form of Poincare recurrence)** Let (X, \mathcal{A}, μ) be a **probability space** and $T : X \rightarrow X$ be a μ -preserving transformation and $B \subseteq X$ such that $\mu(B) > 0$. Then the set of points in B that never returns to B has measure zero. >

- Now consider the set of points in B that do not return to B *infinitely many times* that is

$$A_\infty := \{x \in B \mid \exists k \geq 1 : \forall n > k (T^n(x) \notin B)\}$$

- If we consider the sets

$$A := \{x \in B \mid \forall n > 0 (T^n(x) \notin B)\}$$

$$A_k := \{x \in B \mid T^k(x) \in B \text{ and } \forall n > k (T^n(x) \notin B)\}$$

for $k \geq 1$,

- We see

$$x \in T^{-k}(A_0) \iff T^k(x) \in A$$

$$\iff T^k(x) \in B \text{ and } \forall n > 0 \ T^n(T^k x) \notin B$$

so

$$A_k = T^{-k}(A)$$

for all $k \geq 1$.

- If $x \in A_\infty$ there must be a largest integer k such that $T^k(x) \in B$ so $x \in A_k$. Conversely, if $x \in B \cap A_k$, we know

$$T^n(x) \notin B$$

for all $n > k$ meaning $x \in A_\infty$.

- Thus

$$A_\infty = B \cap \bigcup_{k \in \mathbb{N} \cup \{\infty\}} A_k$$

implying

$$A_\infty \subseteq \bigcup_{k \in \mathbb{N} \cup \{\infty\}} A_k$$

$$\implies \mu(A_\infty) \leq \mu\left(\bigcup_{k \in \mathbb{N} \cup \{\infty\}} A_k\right)$$

- And we have already proven $\{A_k \mid k \in \mathbb{N}\}$ are all disjoint

- Consider the set $A \subset B$ of points that do not return to B

$$A := \{x \in B \mid \forall n > 0 : T^n(x) \notin B\}$$

, so this means points of A do not return to A either. If $T^{-n}(A)$ are *not* pairwise disjoint for all $n \in \mathbb{N}$ then there exists $n, m \in \mathbb{N}, n > m$

$$x \in T^{-n}(A) \cap T^{-m}(A) \implies T^n(x) \in A, T^{m-n}(T^n(x)) = T^m(x) \in A$$

meaning the point $T^n(x)$ in A returns to A , which contradicts the definition of A . Thus $T^{-n}(A)$ are all disjoint.

- Thus we can conclude by the previous argument

$$\mu(A_k) = 0 \implies \mu(A_\infty) \leq \sum_{k \in \mathbb{N} \cup \{\infty\}} \mu(A_k) = 0$$

(infinite Poincare recurrence) Let (X, \mathcal{A}, μ) be a **probability space** and $T : X \rightarrow X$ be a μ -preserving transformation and $B \subseteq X$ such that $\mu(B) > 0$. Then the set of points in B that returns to B **only finitely many times** has measure zero.