

## Pre-compact subsets of $(\text{EndTop}(X, Y), \mathfrak{T}_{\text{cpt}})$

Let  $X$  be a topological space,  $(Y, d)$  be a metric space, and  $\mathcal{F} \subseteq \text{EndTop}(X, Y)$  be a family of continuous maps

$$\mathcal{F} \ni f : X \rightarrow Y$$

### Definition. Equicontinuous families in $\text{EndTop}(X, Y)$

Let  $X$  be a topological space,  $(Y, d)$  be a metric space, and  $\mathcal{F} \subseteq \text{EndTop}(X, Y)$ . Then

- $\mathcal{F}$  is **equicontinuous at  $x_0 \in X$**  if  $\forall \epsilon > 0$  there exists an open nb  $x_0 \in U \subseteq X$  such that

$$\begin{aligned} \forall x \in U, \forall f \in \mathcal{F} \quad d(f(x), f(x_0)) < \epsilon \\ \iff \forall z \in \mathcal{F}(U), d(z, f(x_0)) < \epsilon \end{aligned}$$

- $\mathcal{F}$  is **equicontinuous** if it is equicontinuous at all points  $x \in X$ .

## equicontinuous and pointwise precompact families are pre-compact

**(weak Arzela-Ascoli theorem onto metric spaces)** Let  $X$  be a topological space,  $(Y, d)$  be a metric space, and

$$\mathcal{F} \subseteq (\text{EndTop}(X, Y), \mathfrak{T}_{\text{cpt}})$$

If  $\mathcal{F}$  is equicontinuous and pointwise precompact then  $\mathcal{F}$  is **pre-compact**.  
If  $X$  is locally compact and Hausdorff, then converse also holds.

## equicontinuous and pointwise precompact are sequentially pre-compact on $\sigma$ -compact and locally compact spaces

**(sequential Arzela-Ascoli theorem on compact spaces onto metric spaces)** Let  $X$  be **compact** and  $(Y, d)$  be a metric space. Then for a equicontinuous and pointwise bounded

$$\mathcal{F} \subseteq (\text{EndTop}(X, Y), \mathfrak{T}_{\text{cpt}})$$

any sequence in  $\mathcal{F}$  always has a subsequence in  $\mathcal{F}$  that converges uniformly on  $X$  to a continuous map  $X \rightarrow Y$ .

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**(weak Arzela-Ascoli theorem onto metric spaces)** Let  $X$  be a topological space,  $(Y, d)$  be a metric space, and

$$\mathcal{F} \subseteq (\text{EndTop}(X, Y), \mathfrak{T}_{\text{cpt}})$$

If  $\mathcal{F}$  is equicontinuous and pointwise precompact then  $\mathcal{F}$  is **pre-compact**.  
If  $X$  is locally compact and Hausdorff, then converse also holds.

and

**Proposition:** If  $X$  is *compact* and  $(Y, d)$  is any metric space, then the compact-open topology on  $Y^X$  and the topology of uniform convergence coincides

$$\mathfrak{T}_{\text{cpt}} = \mathfrak{T}_{\text{uniform}}$$

☰ **(sequential Arzela-Ascoli theorem on  $\sigma$ -compact and locally compact spaces onto metric spaces)** Let  $X$  be a  $\sigma$ -compact and locally compact topological space and  $(Y, d)$  be a metric space. Then a **equicontinuous and pointwise bounded** subset

$$\mathcal{F} \subseteq (\text{EndTop}(X, Y), \mathfrak{T}_{\text{cpt}})$$

is **sequentially pre-compact** (any sequence in  $\mathcal{F}$  always has a subsequence in  $\mathcal{F}$  that converges uniformly on compact subsets of  $X$  continuous map  $X \rightarrow Y$ ).