Isometric action on a metric space

Definition. Distance on the quotient space of isometric action on a metric space

Let (X, d_X) is a metric space and $\Gamma \leq \text{Isom}(X)$ be a group acting by isometries on X. Then the quotient space

$$X \to {}_{\Gamma}\backslash^{X}$$

has the distance

$$\begin{aligned} d_\Gamma : {}_{\Gamma}\backslash^{X} \times {}_{\Gamma}\backslash^{X} \to [0, \infty] \\ d_\Gamma(\Gamma\{x\}, \Gamma\{y\}) := \inf\{d(\gamma x, \gamma y) | x, y \in X\} \end{aligned}$$

Proposition:

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is a metric $\iff$ each Γ-orbit of X is a closed set.

In this case, the d_Γ-topology agrees with the quotient topology: for each $x \in X, r > 0$ we have

$$\Gamma B_r^{d_X}(x) = B_r^{d_\Gamma}(\Gamma\{x\})$$

fundamental domains

Definition. Fundamental domain of an isometric action

Let (X, d_X) is a metric space. A closed subset $F \subseteq X$ is a **(closed) fundamental domain** for $\Gamma \leq \text{Isom}(X)$ if

- $X = \bigcup_{\gamma \in \Gamma} \gamma(F) \iff F \to {}_{\Gamma}\backslash^{X}$ is surjective
- $\left\{\gamma(\mathring{F}) \middle| \gamma \in \Gamma\right\}$ is a pairwise disjoint collection of open sets $\iff \mathring{F} \hookrightarrow {}_{\Gamma}\backslash^{X}$ is injective

Proposition: A fundamental domain exists $\implies$ orbits are discrete.

Definition. Locally finite fundamental domains

Let (X, d_X) is a metric space. A (closed) fundamental domain $F \subseteq X$ for $\Gamma \leq \mathrm{Isom}(X)$ is **locally finite** if $\{\gamma(F) | \gamma \in \Gamma\}$ is a locally finite collection of closed sets.

Let (X, d_X) is a metric space and F be a fundamental domain for $\Gamma \leq \mathrm{Isom}(X)$. Consider the commutative diagram

$$\begin{array}{ccc} F & \overset{\iota}{\hookrightarrow} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ {}^{F}/_{\Gamma} & \overset{\overline{\iota}}{\to} & {}^{X}/_{\Gamma} \end{array}$$

The continuous bijection $\overline{\iota}$ is a homeomorphism $\iff$ F is a locally finite fundamental domain.

Let F be a locally finite fundamental domain.

- Let $U \subseteq {}^{F}/_{\Gamma}$ be open. As π' is continuous $\pi'^{-1}(U) \subseteq F$ is open which means there is a open $V \subseteq X$ such that

$$\pi'^{-1}(U) = F \cap V$$

- Consider

$$\begin{aligned} W &:= \bigcup_{\gamma \in \Gamma} \gamma(F \cap V) \\ \implies \pi(W) &= \pi(F \cap V) \\ &= \overline{\iota}\pi'(F \cap V) \\ &= \overline{\iota}(U) \end{aligned}$$

- **Intuition**

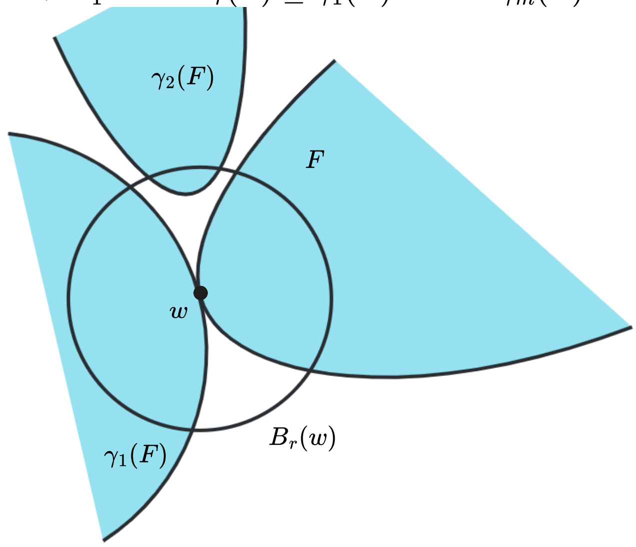
  We must show W is open ($\implies$ $\pi(W)$ is open as π is open $\implies$ $\overline{\iota}$ is an open map $\implies$ $\overline{\iota}$ is a homeomorphism). However

$$\overline{\iota}|_{\mathring{F}/_{\Gamma}} = \underbrace{\pi}_{\text{open}} \circ \underbrace{\iota|_{\mathring{F}}}_{\text{open}} \circ \underbrace{\pi'^{-1}|_{\mathring{F}/_{\Gamma}}}_{\text{open}}$$

  is therefore open. Hence, we must check open-ness on the boundary ∂F

- Let $\gamma w \in W$ be arbitrary where $w \in F \cap V$. As F is locally finite, there is a $r_1 > 0$ such that $B_r(w)$ for $r > r_1$ intersects only finitely many $\gamma(F)$, say, $\gamma_1(F), \ldots, \gamma_m(F)$, so we have

$$r > r_1 \implies B_r(w) \subseteq \gamma_1(F) \cup \cdots \cup \gamma_m(F)$$



- If $\gamma_i(F)$ does not contain w then

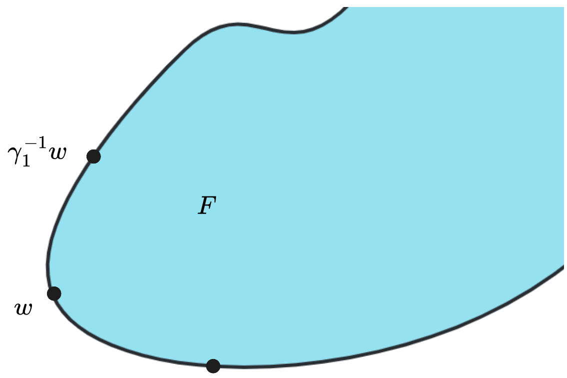
$$\underbrace{B_r(w) \setminus \underbrace{\gamma_i(F)}_{\text{closed}}}_{\text{open}}$$

  is a onb of w that does not intersect $\gamma_i(F)$. Therefore there is a ball at w inside $B_r(w) \setminus \gamma_i(F)$.
- By shrinking r_1 to a smaller $r_2 > 0$ we have

$$r > r_2 \implies B_r(w) \subset \bigcup_{w \in \gamma_i(F)} \gamma_i(F)$$

- Now

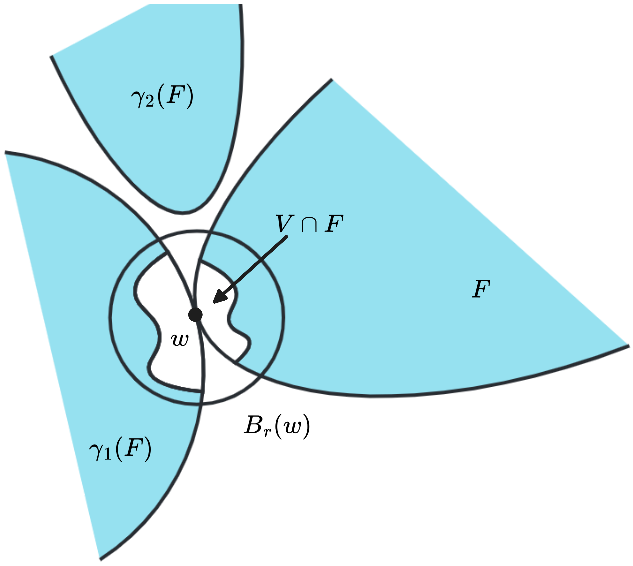
$$\begin{aligned} & w \in \gamma_i(F) \\ \implies & \gamma_i^{-1}(w) \in F \\ \implies & \gamma_i^{-1}(w) \in F \cap V \qquad \text{as } \pi'(\gamma_i^{-1}(w)) = \pi'(w) \in \pi'(F \cap V) \\ \implies & w \in \gamma_i(F \cap V) \end{aligned}$$



- As

$$\bigcup_{w\in\gamma_i(F)} \underbrace{\gamma_i(V)}_{\text{open}}$$

is open, there is a ball near w inside it.



Thus, by shrinking r_2 still further to $r_3 > 0$ we may assume

$$r > r_3 \implies B_r(w) \subseteq \bigcup_{w\in\gamma_i(F)} \gamma_i(V)$$

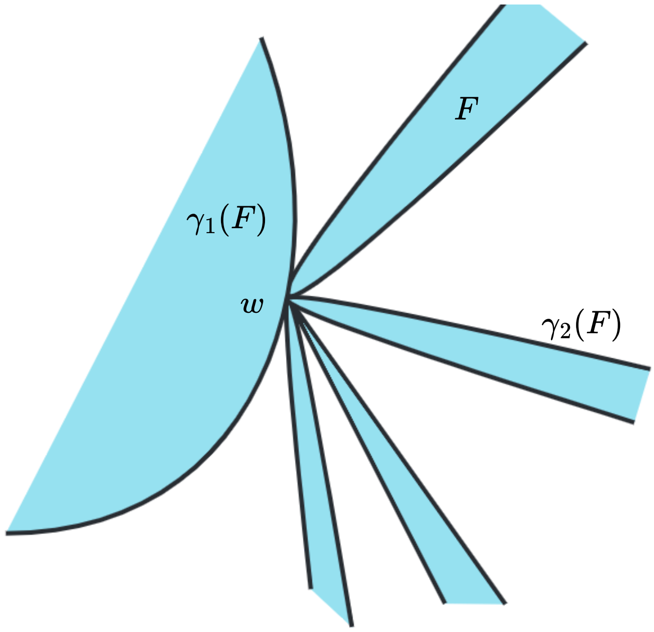
- Therefore,

$$\begin{aligned} r > r_3 \implies B_r(w) &\subseteq \bigcup_{w\in\gamma_i(F)} \gamma_i(V) \cap \bigcup_{w\in\gamma_i(F)} \gamma_i(F) \\ &\subseteq \bigcup_{w\in\gamma_i(F)} \gamma_i(V\cap F) \\ &\subseteq W \end{aligned}$$

so W is open.

Let F is **not** locally finite, that is, there is a $w \in F$ such that

$$\forall \epsilon > 0, \left|\underbrace{\{\gamma\in\Gamma | B_\epsilon(w)\cap\gamma F \neq \emptyset\}}_{F_\epsilon(w)}\right| = \infty$$



Then

$$\gamma_n \in \left\{\gamma \in \Gamma \middle| B_{\frac{1}{n}}(w) \cap \gamma F \neq \emptyset\right\} \setminus \{\gamma_1, \ldots, \gamma_{n-1}\}$$

- **Intuition**

  We must show $\overline{\iota}$ is **not** closed.

- Thus

  $$\gamma_n \underbrace{w_n}_{\in F} \xrightarrow{n\to\infty} w \in F$$

  and for $K := \{w_n\}$ but

  $$\pi(\gamma_n w_n) = \underbrace{\pi(w_n)}_{\in \pi(K)} \xrightarrow{n\to\infty} \pi(w) \notin \pi(K)$$

  So $\pi(K)$ is **not** closed.
- Choose

  $$r := \frac{1}{2} d(x, \Gamma\{y\} \setminus \{x\}) > 0$$

- As γ_n are all distinct and $x = \gamma_n^{-1} w$ for $\gamma_n \in \Gamma_w$ which is finite. Thus

  $$d(w, \gamma_n^{-1} w) \geq 2r$$

- Thus $B_r(x) \cap K$ is finite so there is an open ball at x avoiding K, thus K is closed.
- Now

  $$\pi'^{-1}(\pi'(K)) = \underbrace{K}_{\text{closed}}$$

thus $\pi'(K)$ is closed.

- However

$$\underbrace{\pi'(K)}_{\text{closed}} \overset{\bar{\iota}}{\mapsto} \underbrace{\pi(K)}_{\text{not closed}}$$