Quasi-isometries between metric spaces

Definition. Quasi-isometries between metric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A map

$$f : X \to Y$$

- is a (c, b)-**quasi-isometric embedding** if there are constants $c, b > 0$ such that $\forall x_1, x_2 \in X$

$$\frac{1}{c} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2) + b$$

- is **finite distance** from $f' : X \to Y$ if

$$\exists c > 0 : \forall x \in X, d_Y(f(x), f'(x)) \leq c$$

- has a **quasi-inverse** $g : Y \to X$ if $g \circ f$ has finite distance from Id_X and $f \circ g$ has finite distance from Id_Y
- is a **quasi-isometry** if it is a *quasi-isometric embedding* with a *quasi-inverse* $g : Y \to X$ which is also a quasi-isometry

In general, quasi-isometries are

- not injective
- not surjective
- not continuous
- are not at a finite distance from an isometry
- not dimension preserving

Definition. Category of metric spaces and quasi-isometric embeddings

Let $\mathsf{Met}_{\mathsf{preQ}}$ be the category

- whose objects are metric spaces
- morphisms are quasi-isometric embeddings
- composition of morphisms are given by composition of maps

Definition. Category of metric spaces and quasi-isometric embeddings upto finite distance

Let $\mathsf{Met}_{\mathsf{Q}}$ be the category

- whose objects are metric spaces

- morphisms are quasi-isometric embeddings upto finite distance

$$\text{HomMet}_{\mathsf{Q}} := \frac{\{X \to Y \text{ quasi-isometric embedding}\}}{\text{finite distance}}$$

- composition of two morphisms is the equivalence class of the composition of maps

$$\begin{aligned} \text{HomMet}_{\mathsf{Q}}(Z, Y) \times \text{HomMet}_{\mathsf{Q}}(X, Y) &\to \text{HomMet}_{\mathsf{Q}}(X, Z) \\ ([g], [f]) &\mapsto [g \circ f] \end{aligned}$$

Proposition: Let (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ be a map. Then f is a quasi-isometry $\iff$

- f is quasi-isometric embedding
- f has **quasi-dense image**:

$$\exists c > 0 : \forall y \in Y \exists x \in Y : d_Y(f(x), y) \leq c$$