$$GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$$

> ✈ Definition. **The transvection action** $GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$
>
> The smooth action
>
> $$\boldsymbol{t}: GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$$
> $$p \overset{g}{\mapsto} \boldsymbol{t}_g(p) := gpg^{\mathsf{T}}$$
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> is called transvection.

> **Proposition:** The action
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> > ✈ Definition. **The transvection action** $GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$
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> > is called transvection.
>
> is transitive with stabilizer of $I \in \mathsf{Pd}(n,\mathbb{R})$ being $O(n,\mathbb{R})$.

💡 Diagonalize A using a $O \in SO(n,\mathbb{R})$ so

$$OAO^{\mathsf{T}} = D \implies A = O^{\mathsf{T}}AO$$

now

$$g := O^{\mathsf{T}}\sqrt{D}O$$

means

$$g = g^{\mathsf{T}}, A = gg^{\mathsf{T}}$$

Thus

$$A = gIg^{\mathsf{T}}$$

homogeneous metric

> **Proposition:**
>
> - The action of $GL(n,\mathbb{R})$ is by isometries
>
> $$GL(n,\mathbb{R}) \curvearrowright (\mathsf{Pd}(n,\mathbb{R}), g_{\mathsf{Pd}})$$
>
> - The complete metric space $(\mathsf{Pd}(n,\mathbb{R}), d_{\mathsf{Pd}})$ is CAT(0).
> - The exponential of matrices is the Riemannian exponential
>
> $$\exp = \exp_I$$

- The inclusion

$$GL(n, \mathbb{R}) \to \mathrm{Isom}(\mathsf{Pd}(n, \mathbb{R}), g_{\mathsf{Pd}})$$

  is open? and surjective? onto the connected component of isometry group with kernel

$$\{I, -I\}$$

- The map

$$\sigma_I : \mathsf{Pd}(n, \mathbb{R}) \to \mathsf{Pd}(n, \mathbb{R})$$
$$q \mapsto q^{-1}$$

  is an isometry which fixes I and whose differential at I is $-\mathsf{Id}$. By homogeneity we get a geodesic symmetry at $p \in \mathsf{P}(n, \mathbb{R})$

$$\sigma_p : \mathsf{Pd}(n, \mathbb{R}) \to \mathsf{Pd}(n, \mathbb{R})$$
$$q \mapsto pq^{-1}p$$

  Thus, $(\mathsf{Pd}(n, \mathbb{R}), g_{\mathsf{Pd}})$ **is a CAT(0) symmetric space**.

dimension counts

Fix $n > 1$. Then

$$\dim GL(n, \mathbb{R}) = n^2$$

$$\frac{O(n, \mathbb{R})}{\{I, -I\}} \hookrightarrow O(\mathsf{Sym}(n, \mathbb{R}))$$

whose difference in dimension is

$$\dim O(\mathsf{Sym}(n, \mathbb{R})) = \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right)$$
$$\dim O(n, \mathbb{R}) = \frac{n(n-1)}{2}$$
$$\dim O(\mathsf{Sym}(n, \mathbb{R})) - \dim O(n, \mathbb{R}) = \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right) - \frac{n(n-1)}{2}$$
$$= \frac{n^2(n^2 + 2n - 3)}{4}$$