$SL(2,\mathbb{R})$

#wiki/Man/R/3 #Wiki/group/Lie

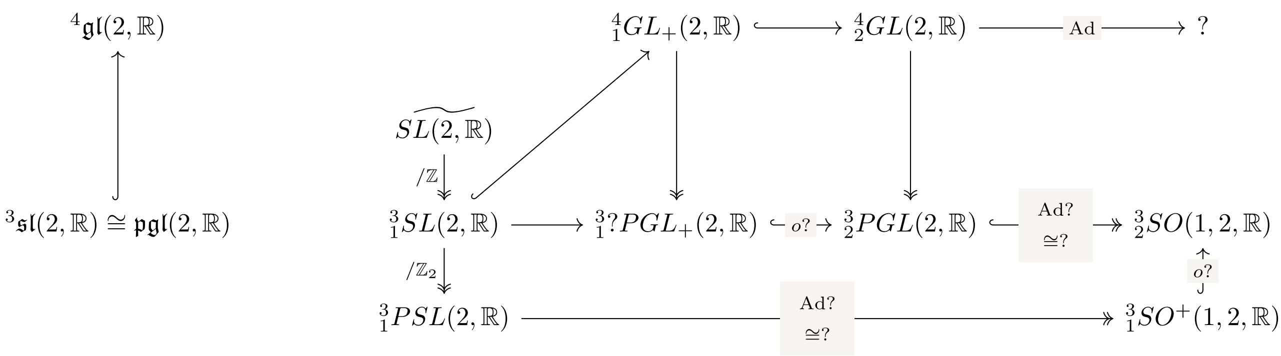
We understand

$$SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$$

by definition.

[1]

subgroups and group extensions



subgroups/extensions	quotient
$SL(2,\mathbb{Z})$	$SL(2,\mathbb{R})/SL(2,\mathbb{Z})$
$\{I,-I\}$	$PSL(2,\mathbb{R})$
$SL(2,\mathbb{C})$	
discrete subgroups $\iff$ preimage of Fuchsian groups	

types

Given a matrix in $SL(2,\mathbb{R})$, it has eigenvalues

$$\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm \frac{\sqrt{\mathrm{tr}^2 - 4}}{2}$$

If the discriminant

- If $\mathrm{tr}^2 - 4 < 0$ then

  $$\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm i\frac{\sqrt{-\mathrm{tr}^2 + 4}}{2}$$

  which has complex norm 1, thus it is a unit complex number, $\lambda \in U(1)$.
- If $\mathrm{tr}^2 - 4 > 0$, then $\lambda \in \mathbb{R}$.

- Let $\lambda \in \{-1, 1\}$ then

$$\operatorname{tr} = \lambda + \frac{1}{\lambda} \in \{-2, 2, 0\}$$

which contradicts $\operatorname{tr}^2 > 4$.
- Thus $\lambda \in \mathbb{R} \setminus \{-1, 1, 0\}$.

Therefore, we have the following cases:

type	**elliptic**	**parabolic**	**hyperbolic**
$\operatorname{tr}^2 - 4$	< 0	$= 0$	> 0
$\lvert \operatorname{tr} \rvert$	< 2	$= 2$	> 2
eigenvalues λ, λ^{-1}	$\lambda \in U(1) \setminus \{-1, 1\}$ $$\lambda, \lambda^{-1} = \frac{\operatorname{tr}}{2} \pm i \frac{\sqrt{-\operatorname{tr}^2}}{2}$$	$$\lambda = \lambda^{-1} = \frac{\operatorname{tr}}{2} \in \{-1, 1$$	$\lambda \in \mathbb{R} \setminus \{-1, 1, 0\}$ $$\lambda, \lambda^{-1} = \frac{\operatorname{tr}}{2} \pm \frac{\sqrt{\operatorname{tr}^2 -}}{2}$$
Jordan block in $GL(2, \mathbb{C})$	$$\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix}$$	$$\pm \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$ for $t \in \mathbb{R}$	$$\pm \begin{bmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{bmatrix}$$ for $x \in \mathbb{R}, \pm e^x = \lambda$
elements in $SL(2, \mathbb{R})$ upto conjugacy	conjugate to rotation matrices $R^\theta \in SO(2, \mathbb{R})$	conjugate to $$\pm \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$ for $t \in \mathbb{R}$	as it has two distinct real eigenvalues so it is conjugate to $$\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$$
$SL(2, \mathbb{R}) \curvearrowright \mathbb{R}^2$	elliptic fixed point at 0	parabolic fixed point at 0	hyperbolic fixed point at 0
$SL(2, \mathbb{R}) \curvearrowright \boldsymbol{P}(\mathbb{R}^2) =: \mathbb{R}$	fixes no line in $\mathbb{R}^2$	fixes exactly one line in $\mathbb{R}^2 \iff$ fixes exactly $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	fixes exactly two lines in $\mathbb{R}^2$
$SL(2, \mathbb{R}) \curvearrowright \mathbb{R}H^2$	fixes one point $\iff$ conjugate to stabilizer of i which is $SO(2, \mathbb{R})$	fixes exactly one point in $\partial \mathbb{R}\boldsymbol{H}^n \iff$ conjugate to a transformation only fixing $\infty \iff$ $$z \mapsto z + t$$	fixes exactly two points in $\partial \mathbb{R}\boldsymbol{H}^n \iff$ conjugate to $$z \mapsto \alpha z$$

exponential map of $SL(2, \mathbb{R})$

det	eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{-}$	type	Jordan block	exponential	trace of exponential
$\det < 0$	$-x, x \in \mathbb{R} \setminus \{0\}$	hyperbolic	$\begin{bmatrix} -x & 0 \\ 0 & x \end{bmatrix}$	$\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{bmatrix}$	$e^{-x} + e^{x} > 0$
$\det = 0$	0	parabolic	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$	2
$\det > 0$	eigenvalues $-ix, ix \in i\mathbb{R}$	elliptic	$\begin{bmatrix} -ix & 0 \\ 0 & ix \end{bmatrix}$	$\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix}$	$2\cos(x) \geq -2$

Hence,

$$\operatorname{tr}(\exp(\mathfrak{sl}(2, \mathbb{R}))) \geq -2$$

So for example

Example

The matrix

$$\begin{bmatrix} -2 & \\ & -\frac{1}{2} \end{bmatrix} \in SL(2, \mathbb{R})$$

is not in the image of the exponential $\mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$.

Let there exists $A \in \mathfrak{gl}(2, \mathbb{R})$ with eigenvalues α, β such that

$$\exp(A) = \begin{bmatrix} -2 & \\ & -\frac{1}{2} \end{bmatrix}$$

Then

$$e^{\alpha} = -2, e^{\beta} = -\frac{1}{2}$$

which implies $\alpha, \beta \in \mathbb{C} \setminus \mathbb{R}$ but because A is a real 2x2 matrix it's eigenvalues must be roots of a quadratic polynomial so

$$\alpha = \bar{\beta} \implies |e^{\alpha}| = \left|e^{\beta}\right|$$

as $|\exp(z)| = e^{\Re(z)}$. But the absolute values do not match $2 \neq \frac{1}{2}$. Thus there is no such $A \in \mathfrak{gl}(2, \mathbb{R})$ either.

generators generating a diffeomorphism

$$
\begin{aligned}
\mathbb{R}\times\mathbb{R}\times S^1 &\to SL(2,\mathbb{R})\\
(x,t,\theta) &\mapsto \begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^t & 0\\ 0 & e^{-t}\end{bmatrix}R^\theta
\end{aligned}
$$

is a diffeomorphism.

[2]

$$(1,0,0)\mapsto I_2$$

$$(1,0,\pi)\mapsto -I_2$$

If we define

$$
\begin{aligned}
X &:= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\
Y &:= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\\
R &:= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}
\end{aligned}
$$

such that

$$
\begin{aligned}
[X,Y] &= 2Y\\
[X,R] &= -2R-4Y\\
[Y,R] &= X
\end{aligned}
$$

so it is not looking that good, but we have

$$
\begin{aligned}
\exp(tX) &= \begin{bmatrix}e^t & \\ & e^{-t}\end{bmatrix}\\
\exp(tY) &= \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\\
\exp(\theta R) &= R^\theta
\end{aligned}
$$

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1. SL(2,R).pdf ↩
2. webspace.science.uu.nl/~ban00101/lecnotes/sltwoR.pdf ↩