$SL(2,\mathbb{R}) \curvearrowright H^2_{\text{U}}$

#Wiki/group-action/Lie

- We observe

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{\Im(z)}{|cz+d|^2}$$

  for $a, b, c, d \in \mathbb{R}$

Definition. $SL(2,\mathbb{R}) \curvearrowright H^2_{\text{U}}$

The group $SL(2,\mathbb{R})$ acts on the upper-half plane H^2_{U} by *Mobius transformations*

$$z \xmapsto{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \frac{az+b}{cz+d}$$

for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$ and $(x,y) \in H^2_{\text{U}}$, where the action by $PSL(2,\mathbb{R})$ is faithful.

> group of complex 1-manifold automorphisms

- Let $x + iy \in H^2$. Then

$$\begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{bmatrix}$$

  maps

$$i \mapsto \frac{\sqrt{y}\, i + \frac{x}{\sqrt{y}}}{\frac{1}{\sqrt{y}}} = x + iy$$

  - Thus the action $SL(2,\mathbb{R}) \curvearrowright H^2_{\text{U}}$ is **transitive**.
- Now, the stabilizer of $i \in H^2_{\text{U}}$ must have

$$|ci+d|^2 = 1$$

  by

  > - We observe
  >
  > $$\Im\left(\frac{az+b}{cz+d}\right) = \frac{\Im(z)}{|cz+d|^2}$$
  >
  > for $a, b, c, d \in \mathbb{R}$

  - That means $(c,d) = (\sin\theta, \cos\theta)$ for some $\theta \in \mathbb{R}$, but

$$\begin{aligned} \frac{ai+b}{\sin\theta\, i + \cos\theta} &= i \\ \implies a, b &= \cos\theta, -\sin\theta \end{aligned}$$

- Thus

$$\mathrm{stab}_{SL(2,\mathbb{R})}(i) = SO(2,\mathbb{R})$$

The transitive, faithful action $PSL(2,\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ by *Mobius transformations* is the connected component of isometry group of the hyperbolic metric on upper-half plane

$$\mathrm{AutRiem}(H^2_{\mathrm{U}}, g_{\mathrm{Hy}})_0 \cong PSL(2,\mathbb{R})$$

Therefore the action is symplectic or area preserving in the hyperbolic area 2-form.

Thus, we have

$$SL(2,\mathbb{R}) \curvearrowright (H^2_{\mathrm{U}}, g_{\mathrm{Hy}}, \lambda_{\mathrm{Hy}})$$

> Lie generators

For the transitive action $SL(2,\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ by *Mobius transformations*, the stabilizer of $i = (0,1) \in H$ is $SO(2,\mathbb{R}) < SL(2,\mathbb{R})$. Thus

$$\frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})} \cong_{\mathsf{Man}} H^2_{\mathrm{U}}$$

by orbit-stabilizer theorem.

sub-actions

- how subgroups act
  - $SL(2,\mathbb{Z}) \curvearrowright H^2_{\mathrm{U}}$
  - Γ torsion-free discrete
  - ...

induced action on $T^{\mathrm{u}} H^2_{\mathrm{U}}$

$$A(z) = \frac{az+b}{cz+d}$$

then

- its *complex* derivative is

$$\begin{aligned} A'(z) &= \frac{a(cz+d) - c(az+b)}{(cz+d)^2} \\ &= \frac{\det A}{(cz+d)^2} \\ &= \frac{1}{(cz+d)^2} \end{aligned}$$

- which is precisely the multiplier for the linear map $\mathfrak{D}_z A$, making the the derivative

$$\begin{aligned} \mathfrak{D}A : TH^2_{\mathrm{U}} &\to TH^2_{\mathrm{U}} \\ (z,v) &\mapsto \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2} \right) \end{aligned}$$

  where $TH^2_{\mathrm{U}} \cong H^2_{\mathrm{U}} \times S^1$ naturally.

- The action preserves the hyperbolic metric because

$$\begin{aligned}\langle D_z A(v), D_z A(w)\rangle_{\text{Hyp}} &= \left(\frac{y}{|cz+d|^2}\right)^{-2}\left(\frac{v}{(cz+d)^2}, \frac{w}{(cz+d)^2}\right)\\ &= \frac{1}{y^2}\langle v, w\rangle_{\text{DOT}}\\ &= \langle v, w\rangle_{\text{Hyp}}\end{aligned}$$

- As the inner product is preserved, hyperbolic unit vectors map to themselves, thus we restrict the action to

$$SL(2,\mathbb{R}) \curvearrowright T^{\mathrm{u}} H^2_{\mathrm{U}}$$

- Now consider the action of the stabilizers

$$A = \frac{i\cos\theta - \sin\theta}{i\sin\theta + \cos\theta}$$

  of $i \in H^2_{\mathrm{U}}$

$$(D_i A)(v) = \frac{v}{(i\sin\theta + \cos\theta)^2} = (\cos 2\theta - i\sin 2\theta)v$$

  - So varying θ, we can map v to any element of $T_i^{\mathrm{u}} H^2_{\mathrm{U}}$.
  - Thus, the action

$$SL(2,\mathbb{R}) \curvearrowright T^{\mathrm{u}} H^2_{\mathrm{U}}$$

  - Now, say A is a stabilizer of (i, v), then

$$\begin{aligned}2\theta \bmod 2\pi &= 0\\ \implies \theta &= \pi\mathbb{Z}\\ \implies A &= \pm I\end{aligned}$$

  - Thus, the action

$$PSL(2,\mathbb{R}) \curvearrowright T^{\mathrm{u}} H^2_{\mathrm{U}}$$

    is transitive and free giving us an isomorphism

$$\begin{aligned}PSL(2,\mathbb{R}) &\cong T^{\mathrm{u}} H^2_{\mathrm{U}}\\ A &\mapsto A(i,i)\end{aligned}$$

Definition. $SL(2,\mathbb{R}) \curvearrowright TH^2_{\mathrm{U}} \cong H^2_{\mathrm{U}} \times S^1$

We have the action

$$\begin{aligned}SL(2,\mathbb{R}) &\curvearrowright TH^2_{\mathrm{U}} \cong H^2_{\mathrm{U}} \times S^1\\ (z, v) &\xrightarrow{\begin{bmatrix} a & b\\ c & d\end{bmatrix}} \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right)\end{aligned}$$

induced by $SL(2,\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ where the action by $PSL(2,\mathbb{R})$ is free and transitive giving us an isomorphism of left $PSL(2,\mathbb{R})$-manifolds

$$\begin{aligned}PSL(2,\mathbb{R}) &\cong T^{\mathrm{u}} H^2_{\mathrm{U}}\\ A &\mapsto A(i,i)\end{aligned}$$

geodesic flow

We have the geodesic flow

$$\exp(t\mathrm{G}^{H^2_{\mathrm{U}}}) : T^{\mathrm{u}}H^2_{\mathrm{U}} \to T^{\mathrm{u}}H^2_{\mathrm{U}}$$

of the hyperbolic metric on H^2_{U}

- We have

$$\exp(t(i,i)) = (e^t i, e^t i) = \mathfrak{D}_i \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}(i)$$

- For the geodesic starting at $(z,v) = A(i,i)$ we have

$$\begin{aligned} \exp(t(z,v)) &= \mathfrak{D}A(\exp(t(i,i))) \\ &= \mathfrak{D}A\left(\mathfrak{D}_i \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}(i)\right) \\ &= \mathfrak{D}_i \left(A \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}^{-1}\right)(i) \end{aligned}$$

- Thus, under the identification

Definition. $SL(2,\mathbb{R}) \curvearrowright TH^2_{\mathrm{U}} \cong H^2_{\mathrm{U}} \times S^1$

We have the action

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induced by $SL(2,\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ where the action by $PSL(2,\mathbb{R})$ is free and transitive giving us an isomorphism of left $PSL(2,\mathbb{R})$-manifolds

$$\begin{aligned} PSL(2,\mathbb{R}) &\cong T^{\mathrm{u}}H^2_{\mathrm{U}} \\ A &\mapsto A(i,i) \end{aligned}$$

the geodesic flow corresponds to right multiplication by $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}^{-1}$ that is

$$A \xrightarrow{t} A \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}^{-1}$$