ⓘ Info

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Continuous group action on a topological space

Definition. Continuous group action on a topological space

A **continuous (left) group action** Φ on a topological space X by a topological group G is a group action

> **Definition.** Group action on a set
>
> $$\Phi : G \curvearrowright S$$
>
> Given a group $(G, \star)$ and a set S. Then the function
>
> $$\begin{aligned} \Phi : G \times S &\to S \\ (g, p) &\mapsto \Phi^g(p) \end{aligned}$$
>
> is called a **(left) group action** $\Phi : G \curvearrowright S$ if
>
> - $$p \in S \implies \Phi^i(p) = p$$
>
>   where i is the identity element of G
> - $$h, g \in G,\ P \in S \implies \Phi^h(\Phi^g(p)) = \Phi^{h \star g}(p)$$

such that the map

$$\Phi : G \times X \to X$$

is continuous.

- Each Φ^g is continuous as the restriction

  $$\{g\} \times X \to X$$

  is continuous. It has a continuous inverse $\Phi^{g^{-1}}$, hence each Φ^g are homeomorphisms.
- Equivalently it is a group homomorphism

  $$G \to \mathrm{Homeo}(X)$$

  such that the map $G \times X \to X$ is continuous.
- orbit space
- discrete group acts by homeomorphisms $\iff$ continuous action by the discrete group
- action by $S \subseteq \mathrm{Homeo}(X)$
- types of actions by arbitrary G
  - faithful
- actions by type of G
  - discrete
  - compact
  - locally compact
  -

orbit space

Definition. Orbit space and projection of a continuous group action

Given a continuous group action $G \curvearrowright X$ we produce the orbit space and projection map π

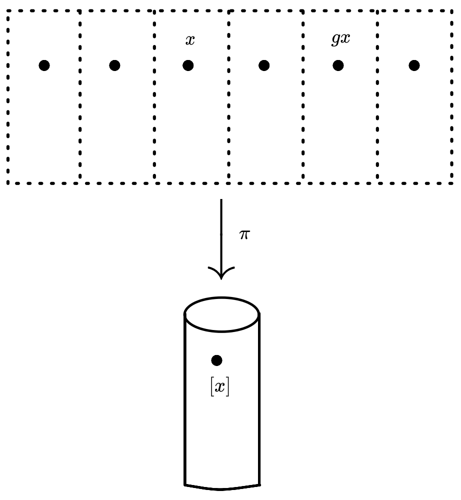
$$G \curvearrowright X \xrightarrow{\pi} \frac{X}{G}$$

by

$$\frac{X}{G} := \frac{X}{x \in X, g \in G \iff x \sim gx}$$

with the quotient topology and quotient projection

$$\begin{aligned} \pi : X &\to \frac{X}{G} \\ x &\mapsto [x] \end{aligned}$$

- 

- The map π is obviously G-invariant as

$$\begin{tikzcd} x \arrow[d, "g"] & \mapsto & {[x]} \\ gx & \mapsto & {[gx] = [x]} \end{tikzcd}$$

- The preimage of projection of $U \subseteq X$ is the obit of U under G

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$$

In general quotient maps are not open or closed, but this formula shows group quotient maps are open:

In

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π is a open map.

For a open $U \subseteq X$ by

- The preimage of projection of $U \subseteq X$ is the obit of U under G

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$$

$\pi^{-1}(\pi(U))$ is union of gU which are open as g are homeomorphisms.
As $\pi^{-1}(\pi(U))$ is open iff $\pi(U)$ is open, π is open.

from a set of homeomorphisms

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Continuous group action on a topological space

And it has 29 siblings.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
    - Discrete subgroup acting on coset space
    - Basis and subbasis for a topology
    - Principal G-bundles over topological spaces

$$G \curvearrowright P \to X$$

    - Vector bundles over topological spaces
    - Closure of a subset of topological space
    - Locally compact and 2-countable Hausdorff topological spaces are paracompact with countable refinement
    - Covering dimension of topological spaces
    - $\mathrm{EndTop}(X, Y)$
    - Eilenberg–Steenrod functors

$$(H_n : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Ab}, \partial_n)$$

    - Local fields
    - Local fields of characteristic 0
    - Topological groups
    - Continuous group action on a topological space
    - Intersection of open, dense subsets of a topological space
    - Local topological spaces
    - Covering space of a topological space
    - Borel measures on a topological space
    - Convergence for nets in a topological space
    - Loops and a open cover of a topological space
    - Paths in a topological space
    - Product of topological spaces
    - Ringed topological spaces
    - Sheaf on open sets of a topological space
    - Closed subsets of a topological space
    - Discrete subsets of topological spaces
    - Types of topological spaces
    - Locally compact Hausdorff spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$