Riemannian manifolds with ergodic geodesic flow

[1]

Corollary 2.2 (Quantum ergodicity for Laplacian eigenfunctions)**.** *Suppose* (M, g) *is a compact Riemannian manifold with ergodic geodesic flow. Then there exists a sequence* $j_k \to \infty$ *which has density 1 in the sense*

$$\lim_{N\to\infty} \frac{\#\{j_k \le N\}}{N} = 1,$$

such that for each $f \in C^\infty(M)$,

$$\int_M |\varphi_{j_k}|^2 f dx \to \frac{1}{\mathrm{Vol}(M)} \int_M f(x)dx. \tag{6}$$

In particular, if we take $U \subset M$ to be an open subset, and f a sequence of smooth functions that approximates the characteristic function χ_U of U, then we will get

$$\int_U |\varphi_{j_k}|^2 dx \to \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(M)}.$$

In other words, for *most* eigenfunctions, the "mass" will tends to be *equidistributed* on M.

In the statement of quantum ergodicity theorems above, one can't rule out the possibility of the existence of *quantum scar*, namely a density 0 subsequence along which the integral (6) concentrate on a closed geodesic (or the corresponding integral (5) will concentrate on a periodic orbit of the geodesic flow)[2]. For the case of negatively curved manifolds (in which case it is known that the geodesic flow satisfy a stronger ergodicity condition), it was conjectured by Z. Rudnick and P. Sarnak in 1994 that there is no quantum scar. This is known as *Quantum Unique Ergodicity Conjecture*, and is currently one of the most famous open problems in spectral geometry:

Conjecture 2.3 (QUE conjecture)**.** *Suppose* (M, g) *is a compact Riemannian manifold with negative sectional curvature. Then as* $k \to \infty$,

$$\int_M |\varphi_k|^2 f dx \to \frac{1}{\mathrm{Vol}(M)} \int_M f(x)dx.$$

[2]The limit can't be too bad. See the theorem at the end of this lecture

1. http://staff.ustc.edu.cn/~wangzuoq/Courses/20F-SMA/Notes/Lec24-25.pdf#page=4.67&gsr=0 ↩