smooth $\mathbb{R}$-manifolds of dimension 2

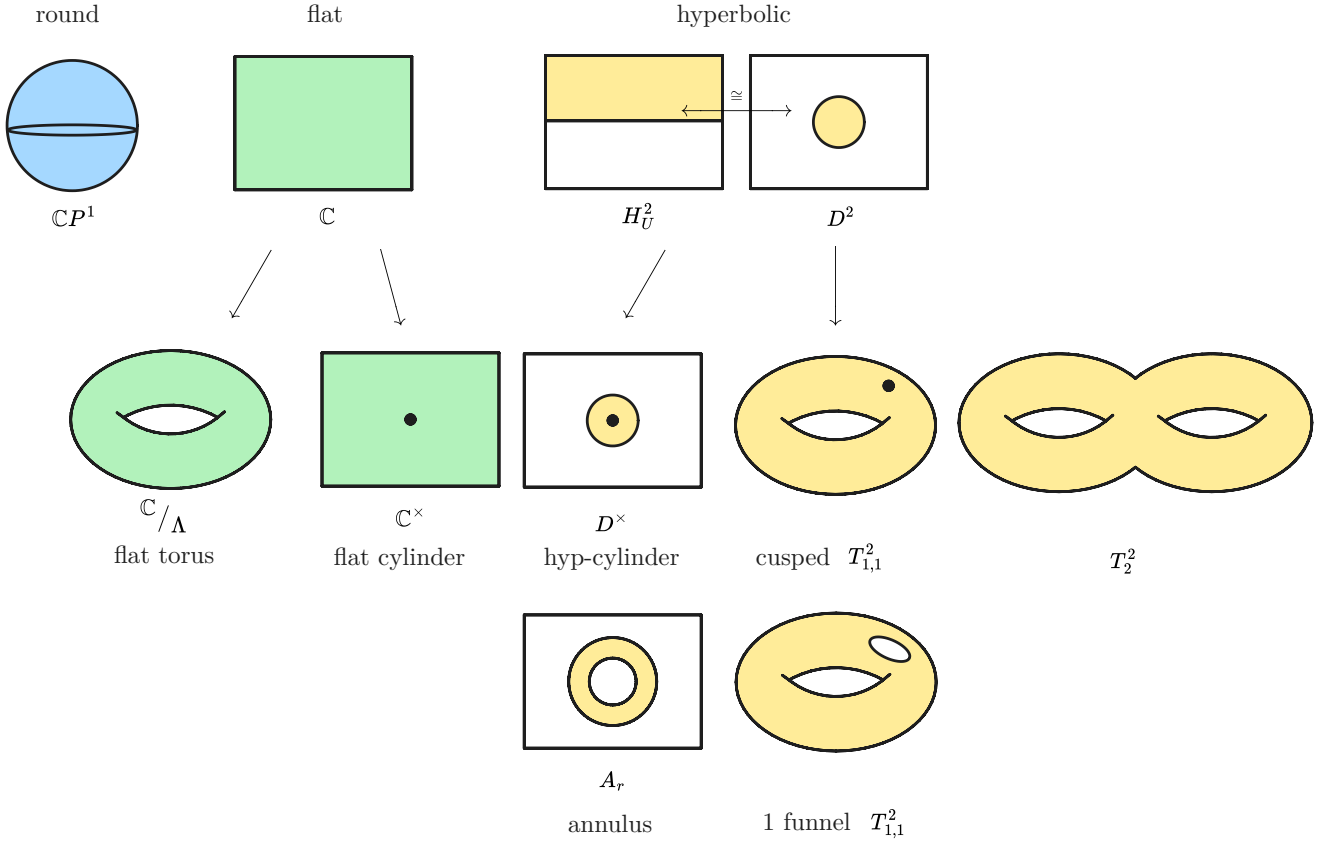
Every orientable smooth $\mathbb{R}$-manifold of dimension 2 has a structure of a $\mathbb{C}$-manifold of dimension 1.

We may study the "classification" of connected, orientable smooth $\mathbb{R}$-manifolds of dimension 2 through the "classification" of Riemann surfaces:

Definition. **Connected complex $\mathbb{C}$-manifolds of $\mathbb{C}$-dimension 1 are called Riemann surfaces.**

(Uniformization of simply connected Riemann surfaces) Every *non-compact simply connected* Riemann surface is biholomprhic to the complex plane $\mathbb{C}$ or the unit disk D. Every *compact simply connected* Riemann surface is biholomorphic to the Riemann sphere $\mathbb{C}P^1$

So every Riemann surface is a quotient of $\mathbb{C}P^1$ or $\mathbb{C}$ or H^2_U by a discrete subgroup of their $\mathrm{AutMan}_{\mathbb{C}}$ group.



$\downarrow g$ $\xrightarrow{n}$	0 (T^2_g is connected sum of g many 2-torus T^2)	1	2	3
0	space.R.sphere.2			
1				
2				
3				

$$\begin{array}{ccc} \mathrm{Teich}(T_{g,n,k}) & & \\ \downarrow & & \\ \mathrm{Mod}(T_{g,n,k}) & \to & \end{array}$$

degree d	genus $g = \frac{(d-1)(d-2)}{2}$	dim of moduli of smooth curves $3g-3$	dim of moduli of smooth curves in $\mathbb{C}P^2$ $\binom{d+2}{2} - 9$	dim of moduli of hyperelliptic Riemann surfaces $2g-1$
3	1	1	1	1
	2	3		3
4	3	6	6	5
	4	9		7
	5	12		9
5	6	15	12	11
	7	18		13
	8	21		15
	9	24		17
6	10	27	19	19
	11	30		21
	12	33		23
	13	36		25
	14	39		27
7	15	42	27	29
	16	45		31

Marked hyperbolic surfaces

Definition. **Category of marked hyperbolic surfaces**

Let S be a smooth 2-manifold (without boundary). Then

$$\mathbb{R}\boldsymbol{H}^2\text{-Riemh}(S)$$

denotes the category where

- **objects** are *marked hyperbolic structures on S*, that is, they are diffeomorphisms

$$\underbrace{(X, g_X)}_{\text{complete hyperbolic 2-manifold}} \xrightarrow{\Phi} S$$

- **morphisms** are isometric embeddings which are homotopic

Then

$$\mathbb{R}\boldsymbol{H}^2\text{-Riemh}(S)_{\cong}$$

is the set of all *marked hyperbolic structures* upto homotopic isometries.

$$\mathbb{R}\boldsymbol{H}^2\text{-Riemh}(S)_{\cong} \leftrightarrow \left\{\Gamma \leq PSL(2,\mathbb{R}) \text{ discrete} \middle|_{\Gamma} \backslash^{\mathbb{R}\boldsymbol{H}^2} \cong_{\text{Man}} S\right\} \Big/ PGL(2,\mathbb{R})$$

$$\text{Manh}_{\mathbb{C}}(S)_{\cong} \leftrightarrow \left\{\Gamma \leq PSL(2,\mathbb{R}) \text{ discrete} \middle|_{\Gamma} \backslash^{\mathbb{R}\boldsymbol{H}^2} \cong_{\text{Man}} S\right\} \Big/ PSL(2,\mathbb{R})$$

$$\mathbb{R}H^2\text{-Riemh}(S)_{\cong} \longleftrightarrow \left\{\Gamma \leq PSL(2,\mathbb{R}) discrete \middle|_{\Gamma} \backslash^{\mathbb{R}H^2} \cong_{\text{Man}} S\right\} \Big/ PGL(2,\mathbb{R})$$