Exponential sum of a discrete subgroup of $\text{Isom}(\mathbb{R}\boldsymbol{H}^n)$

> Let $\Gamma \leq \text{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

Intuition

We are interested in the **rate at which orbit tends to S^{n-1}**.

- Any two orbits $\Gamma(a), \Gamma(b)$ are *comparable*: As

  > **Proposition:**
  >
  > $$d_{\mathbb{R}\boldsymbol{H}^n}(o, x) = \log\left(\frac{1+|x|}{1-|x|}\right)$$

  $$\begin{aligned} & d_{\mathbb{R}\boldsymbol{H}^n}(o, \gamma(b)) \leq d_{\mathbb{R}\boldsymbol{H}^n}(o, \gamma(a)) + d_{\mathbb{R}\boldsymbol{H}^n}(a, b) \\ \implies & \log\left(\frac{1+|\gamma(b)|}{1-|\gamma(b)|}\right) \leq \log\left(\frac{1+|\gamma(a)|}{1-|\gamma(a)|}\right) + d_{\mathbb{R}\boldsymbol{H}^n}(a, b) \\ \implies & \frac{1-|\gamma(a)|}{1-|\gamma(b)|} \leq 2e^{d_{\mathbb{R}\boldsymbol{H}^n}(a,b)} \end{aligned}$$

- A good measure of rate of convergence of $\gamma(a) \xrightarrow{\gamma \in \Gamma} S^{n-1}$ is

  $$\alpha \mapsto \sum_{\gamma \in \Gamma} (1 - |\gamma(a)|)^{\alpha}$$

  for $\alpha > 0$. This convergence is independent of a by above.
- However, it is more natural to consider

  $$\alpha \mapsto \sum_{\gamma \in \Gamma} \exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o, \gamma(o)))$$

  and because

  $$\exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o, \gamma(o))) = \left(\frac{1-|\gamma(o)|}{1+|\gamma(o)|}\right)^{\alpha} \leq (1 - |\gamma(o)|)^{\alpha}$$

  the two series convergence or diverge together for fixed $\alpha > 0$.

- Let $o \in \mathbb{R}\boldsymbol{H}^n$ be fixed. We consider the partial sum

$$\begin{aligned}
&\sum_{\gamma \in \Gamma, d_{\mathbb{R}\boldsymbol{H}^n}(x,\gamma(x))<R} \exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(o))) \\
&= \int_{(0,R)} \exp(-\alpha t) \mathrm{d}N(t,o,o) \\
&= N(R,o,o)e^{-\alpha R} + \alpha \int_{t \in (0,R)} N(t,o,o)e^{-\alpha t} \mathrm{dt}
\end{aligned}$$

Proposition:

> Let $\Gamma \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

Let $\alpha > n-1, x \in X$. Then

$$\sum_{\gamma \in \Gamma} \exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(x,\gamma(x))) < \infty$$

Consider the partial sum

- Let $o \in \mathbb{R}\boldsymbol{H}^n$ be fixed. We consider the partial sum

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\end{aligned}$$

and by

> **Let**
>
> $$\Gamma \leq O^{+}(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> **be a discrete subgroup.**

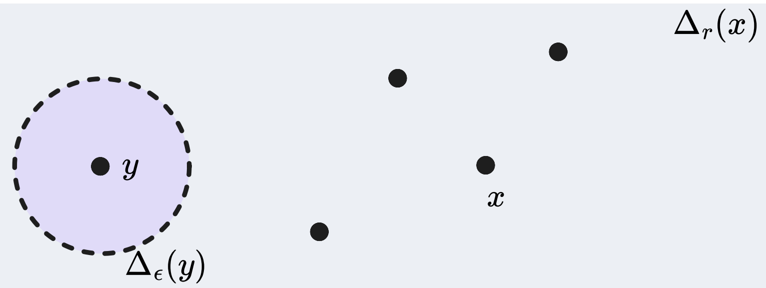
Let $y \in \mathbb{R}\boldsymbol{H}^n$. Then there is a $A_{\Gamma,y} > 0$ such that

$$\forall x \in \mathbb{R}\boldsymbol{H}^n, N(r,x,y) \leq A_{\Gamma,y} e^{r(n-1)}$$

Let $\epsilon > 0$ be small enough so that

$$\{\gamma(\Delta_\epsilon(y)) | \gamma \in \Gamma\}$$

do not overlap unless $\gamma \in \Gamma_y$



by proper

- Then by the volume of balls

$$\begin{aligned}\mu_{\mathbb{R}\boldsymbol{H}^n}(B_s(o)) = m_{S^{n-1}}(S^{n-1}) & \int_{t\in(0,s)} (\sinh t)^{n-1} \mathrm{d}t \\ \leq \frac{m_{S^{n-1}}(S^{n-1})}{2^{n-1}} & \int_{t\in(0,s)} e^{(n-1)t} \mathrm{d}t \\ \leq \frac{m_{S^{n-1}}(S^{n-1})}{2^{n-1}} & \left(\frac{e^{(n-1)s}}{(n-1)} - 1\right)\end{aligned}$$

we have

$$\begin{aligned}\bigsqcup_{\gamma\in\Gamma:\gamma(y)\in B_r(x)} \gamma(\Delta_\epsilon(y)) &\subset \Delta_{r+\epsilon}(x) \\ \mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_\epsilon) N(r,x,y) &\leq \mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_{r+\epsilon}(x)) \\ N(r,x,y) &\leq \frac{\mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_{r+\epsilon})}{\mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_\epsilon)} \\ &= \left(\frac{e^{(n-1)(r+\epsilon)}}{(n-1)} - 1\right)\end{aligned}$$

we have

$$\begin{aligned}\alpha > n-1 \implies & \int_{t\in(0,R)} A_{\Gamma,o} e^{t(n-1)-\alpha t} \mathrm{d}t \\ & = A_{\Gamma,o} \frac{1}{n-1-\alpha} (e^{R(n-1-\alpha)} - 1) \\ & \xrightarrow{R\to\infty} < \infty\end{aligned}$$

Definition. Critical exponent of discrete subgroup

> Let $\Gamma \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

$$\delta_\Gamma := \inf \left\{ \alpha \middle| \sum_{\gamma \in \Gamma} \exp(-\alpha \rho(x, \gamma(x))) < \infty \right\}$$
$$\in [0, n-1]$$

Proposition:

> Let $\Gamma \leq \operatorname{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

If Γ has finite covolume, then

$$\sum_{\gamma \in \Gamma} \exp(-\alpha d(x, \gamma(x)))$$

diverges for $\alpha = n-1$, so $\delta_\Gamma = n-1$.

Consider the partial sum

- Let $o \in \mathbb{R}\boldsymbol{H}^n$ be fixed. We consider the partial sum

$$\begin{aligned} &\sum_{\gamma \in \Gamma, d_{\mathbb{R}\boldsymbol{H}^n}(x, \gamma(x)) < R} \exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o, \gamma(o))) \\ &= \int_{(0,R)} \exp(-\alpha t) \mathrm{d}N(t, o, o) \\ &= N(R, o, o) e^{-\alpha R} + \alpha \int_{t \in (0,R)} N(t, o, o) e^{-\alpha t} \mathrm{d}t \end{aligned}$$

Now by

- Thus for $r > 0$

$$N(r + r_0, o, y_0) \geq A' e^{(n-1)r}$$

we have

$$\begin{aligned} \sum_{\gamma \in \Gamma} \exp(-(n-1) d(x, \gamma(x))) \geq{}& N(R, o, o) e^{(n-1)R} \\ &+ (n-1) A' \int_{t \in (0,R)} \cancelto{1}{e^{(n-1)t} e^{-(n-1)t}} \mathrm{d}t \\ &\geq N(R, o, o) e^{(n-1)R} + (n-1) A' R \\ &\xrightarrow{R \to \infty} \infty \end{aligned}$$